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The Variational Methods for Solving Random Models

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***Abstract*: This paper studies the solutions of variational methods for random ordinary (partial) di□erential equations in L2−space. These methods are called Galerkin method, Petrov-Galerkin method, Least-Squares method and Collocation method. Some basic properties of these methods where applying on random problems will be shown throughout some numerical examples.**

***Keywords*: Random models; Random variational methods, Second order random variable**

I. INTRODUCTION

Deriving the random governing dynamics of physical processes is a complicated task in itself; finding exact solutions to the governing random ordinary (partial) differential equations is usually even more formidable. When trying to solve such equations, approximate methods of analysis provide a convenient, alternative method for finding solutions [1,2,3,4,5,6] Four such methods, the Galerkin method, Petrov-Galerkin method, Least-Squares method and Collocation method, are typically used in our paper and are referred to as classical variational methods [7,8,9,10,11,12,13]. Various variational methods differ from each other in the choice of integral form, weighting functions, and/or approximating functions [7,11].

Classical variational methods suffer from the disadvantage of the difficulty associated with proper construction of the approximating functions for arbitrary domains. This paper is organized as follows, Section 2, deals with preliminaries of some points used in the paper. Section 3, deals with solving a random ordinary and partial differential equation using the variational methods. Section 4, deals with the convergence of the approximation solutions. Section 5 is devoted to some numerical examples.

II. PRELIMINARIES

Definition1. [14] A Hilbert space H is a real or complex inner product space that is also complete metric space with respect to the distance function induced by the inner product,

$$\|f\| = \sqrt{\langle f.f\rangle}$$

An example of a Hilbert space is the L_2 −space which is the space of all functions $f : R \to R$ such that, the integral of f^2 over the whole real line is finite,

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$$

Hilbert spaces constitute the class of infinite-dimensional vector spaces that are most often used and that are the most important as far as applications are concerned. They are the natural extension of the concept of a finite-dimensional vector space with a scalar product.

Definition2.[15] A real random variable X on a probability space (Ω, $\hat{o}$, P) and satisfying the property that

$$E\left[|X|^2\right] < \infty$$

is called second order random variable (2-r.v.) where, $E[\ ]$ denotes the expectation value operator. If $X \in L_2(\Omega)$, then the L_2 norm is defined as

$$\|X\|_2 = \left[E\left[|X|^2\right]\right]^{\frac{1}{2}}.$$

Definition3.[16] The closure of a set S is the smallest closed set containing S i.e., the set of all points of closure of S.

Definition4.[17] Let X be an inner product space, a finite or infinite family of functions $\{\psi_j\}_{j=1} \subset X$ is called an orthogonal set(denoted by $\psi_i \perp \psi_j$) if $(\psi_i, \psi_j) = 0$ when i ≠ j, and $\psi_j \neq 0$. $\{\psi_j\}$ is called an orthonormal set if it is orthogonal and $\|\psi_j\| = 1$.

Definition5.[17] The set of functions $\{\psi_j\}_{j=1}^{n}$ is said to be linearly independent on [a,b] if

$$c_1\psi_1 + c_2\psi_2 + \cdots + c_n\psi_n + = 0 \quad for\, all$$
$$x \in [a,b], c_1 = c_2 = \cdots = c_n = 0.$$

Proposition1.[17] If the set $\{\psi_j\}_{j=1}^{n}$ are non-zero pairwise orthogonal functions, then they are linearly independent.

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III. RANDOM GALERKIN AND PETROV-GALERKIN METHOD

A. Random Galerkin and Petrov-Galerkin Method for an Ordinary Differential Equation

In the traditional variational methods, we seek an approximation solution in the form

$$U(x) \approx U_n(x) = \psi_0 + \sum_j^n c_j \psi_j(x),\ \text{j}:1,\cdots,\text{n} \quad (1)$$

where U_n is the approximation solution, c_j are the random undetermined coefficients and ψ_j are the approximation functions of position x in the domain of the problem, the random differential equation

$$\begin{cases} -\dfrac{d^2U}{dx^2} - \beta U + x^2 = 0 \quad [0,L]\times\Omega \\ U(0) = M_0,\ U(L) = Q_0 \end{cases} \quad (2)$$

β is a second order bounded random variable, M_0 and Q_0 are the known data of the problem. By substituting (1) into (2), we get the non-zero random residual function

$$R(x,c_j) = -\frac{d^2U_n(x)}{dx^2} - \beta U_n(x) + x^2 \neq 0, \text{j}=1,\cdots,\text{n} \quad (3)$$

Remark1.[18] An integrable function w is called a weight function on the interval I if $w(x) \geq 0$ for all $x \in I$, but $w(x) \neq 0$ on any subinterval of I.

We will construct the uniform randomly weighted-integral statement that is equivalent to (2) without including the boundary conditions by multiplying the random residual R by a weight function(i.e., test function) and integrating over the domain:

$$\int_0^L w R\, dx = 0$$

$$\int_0^L w(x)\left(-\frac{d^2U_n(x)}{dx^2} - \beta U_n(x) + x^2\right) dx = 0 \quad (4)$$

$w(x)$ is the weight function and its choice differs from one method to another of the random weighted-residual methods(WRM). (4) is called the random weighted-integral statement that allows us to obtain n linearly random algebraic equations for obtaining the random coefficients c_j. In the weighted-residual methods, we choose ψ_0 so that satisfies the actual boundary conditions of the problem and ψ_i so that satisfies the homogeneous form of specified boundary conditions. Galerkin approach enables us to choose $w_i(x) = \psi_i$, but in the Petrov-Galerkin method $w_i(x) \neq \psi_i$ (i.e., w_i is chosen to be different from ψ_i)

B. Random Galerkin and Petrov-Galerkin Method for A partial Differential Equation

Consider the following random one-dimensional heat problem:

$$\begin{cases} \dfrac{\partial U(x,t)}{\partial t} = \dfrac{\partial^2 U(x,t)}{\partial x^2} & \text{in } R \times T \times \Omega \\ U(x,0) = U_0(x) & \text{in } R \times \Omega \\ U(0,t) = M_0,\ U(L,t) = Q_0 & \text{in } \partial R \times T \times \Omega \end{cases} \quad (5)$$

where U_0 is function of a bounded random variable and M_0, Q_0 are the data of the problem. The approximate solution will be in the form:

$$U(x,t) \approx U_n(x,t) = \psi_0 + \sum_j^n c_j(t)\psi_j(x),\ j:1,\cdots,n$$

and the residual will be in the form:

$$R = \frac{\partial U_n(x,t)}{\partial t} - \frac{\partial^2 U_n(x,t)}{\partial x^2} \neq 0$$

The weighted-integral statement that is equivalent to (5) without including the boundary conditions is in the form (let $x \in [0,L]$, where L is the length of the rod):

$$\int_0^L w(x)\left(\frac{\partial U_n(x,t)}{\partial t} - \frac{\partial^2 U_n(x,t)}{\partial x^2}\right) dx = 0 \quad (6)$$

(6) Enables us to obtain a solvable system of random ordinary differential equations.Substituting the approximation solution

$$U_n(x,t) = \psi_0 + \sum_j^n c_j(t)\psi_j(x)$$

into (6); choosing ψ_0 so that satisfies the actualboundary conditions, taking $w_i = \psi_i$ (Galerkin Approach):

$$\int_0^L \psi_i(x)\left(\frac{\partial}{\partial t}\left[\psi_0 + \sum_j^n c_j(t)\psi_j(x)\right] - \frac{\partial^2}{\partial x^2}\left[\psi_0 + \sum_j^n c_j(t)\psi_j(x)\right]\right) dx = 0$$

$$\frac{\partial}{\partial t}\sum_j^n c_j(t)\int_0^L \psi_i(x)\psi_j(x)dx - \int_0^L \psi_i(x)\frac{\partial^2}{\partial x^2}\psi_0(x)dx$$
$$-\sum_j^n c_j(t)\int_0^L \psi_i(x)\frac{\partial^2}{\partial x^2}\psi_j(x)dx = 0$$

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$$A\,\bar{c}' = F\,\bar{c} + K_i \quad (7)$$

where

$$A_{ij} = \int_0^L \psi_i(x)\psi_j(x)dx$$

$$F_{ij} = \int_0^L \psi_i(x)\frac{\partial^2}{\partial x^2}\psi_j(x)dx$$

$$K_i = \int_0^L \psi_i(x)\frac{\partial^2}{\partial x^2}\psi_0(x)dx$$

$$\bar{c}(t) = \left(c_1(t) \quad c_2(t) \quad \cdots \quad c_n(t)\right)^T$$

$$\bar{c}'(t) = \left(\frac{\partial}{\partial t}c_1(t) \quad \frac{\partial}{\partial t}c_2(t) \quad \cdots \quad \frac{\partial}{\partial t}c_n(t)\right)^T$$

In case of Petrov-Galerkin method, we also choose ψ_0 so that satisfies the actual boundary conditions, and taking $w_i \neq \psi_i$.

C. Random Least-Squares and Collocation Methods

In the Least-Squares method, Minimizing the integral square of the random residual is the way to determine the random coefficients c_j , we have:

$$\int_0^L w(x)R dx = 0$$

$$\frac{\partial R}{\partial c_i}\int_0^L R^2 dx = 0$$

$$\int_0^L \frac{\partial R}{\partial c_i}R(x,c_j)dx = 0 \quad (8)$$

In the random collocation method, the random residual is required to vanish at n selected points (collocation points), we have

$$R(x_i,c_j) = 0, \text{ for } x_i \text{ selected points.} \quad (9)$$

IV. CONVERGENCE OF THE RANDOM APPROXIMATION SOLUTIONS [19,20,21]

A. Convergence of the Random Galerkin and Petrov-Galerkin Solutions

We are eventually solving a matrix equation, either linear as the differential equation

$$\{A\}\{U\} = \{f\}$$

or nonlinear as

$$\{A\}\{U\}\{U\} = \{f\}$$

Where A is a linear and symmetric positive-definite operator defined in a Hilbert space H (i.e., L_2-space), u is the unknown function and f is the known function. A set of functions $\{\psi_n\}$ is said to be linearly independent if

$$\sum_i c_i\psi_i = 0$$

for $c_i = 0$, then one function can be written in terms of the other functions. Completeness of the trial function is a very important property which is satisfied in a space if any function in that space can be expanded in terms of the set of functions

$$\|U - \sum_i c_i\psi_i\| < \varepsilon$$

It is necessary to verify the completeness in a specified space, so the class of functions $\{\psi_n\} \in D(A)$, where $D(A)$ is the domain of definition. We state without proof two important theorems for the mean square convergence of the solutions.

Theorem1. Suppose we have $\{\psi_n\} \in D(A)$, and $A\psi_n$ is complete in a given Hilbert space H . Then, $\{\psi_n\}$ is complete in in L_A^2-space that is obtained by the closure of the domain $D(A)$

Theorem2. If the system coordinate of functions ψ_n is an orthonormal, complete in L_A^2 ,and the random Galerkin and Petrov-Galerkin coefficients are bounded, then the approximation solution (1) is convergent in mean square.

B. Convergence of the Random Least-Squares Solution

We mentioned earlier that Minimizing the functional

$$\int_0^L \left(A(U) - f\right)^2 dx = 0$$

is the core of the Least-Squares method.

Theorem3. If the system coordinate of functions ψ_n is complete in L_A^2 and there exist a constant k such that for any U in the field of definition of the operator A

$$\|U\| \le K\|AU\|$$

, then the approximation solution of the Least-Squares method is convergent in mean square where the random least squares coefficients are bounded.

V. CASE STUDIES

Example 1. The following 1-d boundary value problem for linear ordinary random differential equation:

$$\begin{cases} -\dfrac{d^2U}{dx^2} - \beta U + x^2 = 0 \quad [0,1]\times\Omega \\ U(0) = 0, \; U'(1) = 1 \end{cases} \quad (10)$$

where β is a positive second order bounded random variable.

The exact solution

$$U(x) = \frac{\sin\left(\sqrt{\beta}x\right)\left(\beta^{3/2} + 2\sin\left(\sqrt{\beta}\right) - 2\sqrt{\beta}\right)}{\beta^2\cos\left(\sqrt{\beta}\right)} + \frac{2\cos\left(\sqrt{\beta}x\right) + \beta x^2 - 2}{\beta^2}$$

(11)

Numerical Solution using the random Galerkin Method

Starting from the random weighted-integral statement (4), we choose ψ_0 so that satisfies the actual boundary conditions $\psi_0(0) = 0$ and $\psi_0'(1) = 1$ and ψ_i so thatsatisfies the homogeneous form of specified boundary conditions $\psi_i(0) = 0$ and $\psi_i'(1) = 0$. Now, for $n = 1$, Choosing

$\psi_0 = x$ and $\psi_1 = 2x - x^2$ and taking $w_1 = \psi_1 = 2x - x^2$ (Galerkin approach, $w_i = \psi_i$), we have

$$\int_0^1 \left(2x - x^2\right)\left(-\frac{d^2}{dx^2}\left[x + c_1\left(2x - x^2\right)\right] - \beta\left[x + c_1\left(2x - x^2\right)\right] + x^2\right)dx = 0$$

$$\int_0^1 \left(2x - x^2\right)\left(2c_1 - \beta\left(x + c_1\left(2x - x^2\right)\right) + x^2\right)dx = 0$$

$$\frac{3}{10}c_1\beta - \frac{5}{12}\beta(2c_1 + 1) + \frac{4}{3}c_1 + \frac{3}{10} = 0$$

$$c_1 = -\frac{1}{16}\frac{25\beta - 18}{2\beta - 5}$$

$$U_1(x) = \psi_0 + c_1\psi_1(x) = x - \frac{1}{16}\frac{25\beta - 18}{2\beta - 5}(2x - x^2)$$

For $n = 2$, choosing $\psi_0 = x$, $\psi_1 = 2x - x^2$ and,

$\psi_2 = x^2 - \frac{2}{3}x^3$ and taking $w_i = \psi_i$, $i = 1, 2$, we have

$$\int_0^1 \left(2x - x^2\right)\left(-\frac{d^2}{dx^2}\left[x + c_1(2x - x^2) + c_2\left(x^2 - \frac{2}{3}x^3\right)\right] - \beta\left[x + c_1(2x - x^2) + c_2\left(x^2 - \frac{2}{3}x^3\right)\right] + x^2\right)dx = 0$$

$$\int_0^1 \left(x^2 - \frac{2}{3}x^3\right)\left(-\frac{d^2}{dx^2}\left[x + c_1(2x - x^2) + c_2\left(x^2 - \frac{2}{3}x^3\right)\right] - \beta\left[x + c_1(2x - x^2) + c_2\left(x^2 - \frac{2}{3}x^3\right)\right] + x^2\right)dx = 0$$

Generating two necessary and sufficient random algebraic equations to determine the random coefficients c_1 and c_2 , we get

$$\frac{7}{45}c_2\beta - \frac{3}{10}\beta(c_2 - c_1) + \frac{1}{3}c_2 - \frac{5}{12}\beta(2c_1 + 1) + \frac{4}{3}c_1 + \frac{3}{10} = 0$$

$$\frac{1}{21}c_2\beta - \frac{4}{45}\beta(c_2 - c_1) + \frac{2}{15}c_2 - \frac{7}{60}\beta(2c_1 + 1) + \frac{1}{3}c_1 + \frac{4}{45} = 0$$

$$U_2(x) = \psi_0 + c_1\psi_1(x) + c_2\psi_2(x)$$

$$= x - \frac{\left(39\beta^2 - 1838\beta + 1176\right)(2x - x^2)}{2\left(65\beta^2 - 1692\beta + 3780\right)}$$

$$- \frac{21\left(11\beta^2 - 112\beta + 100\right)(x^2 - \frac{2}{3}x^3)}{2\left(65\beta^2 - 1692\beta + 3780\right)}$$

For $n = 3$,choosing $\psi_0 = x$, $\psi_1 = 2x - x^2$, $\psi_2 = x^2 - \frac{2}{3}x^3$ and $\psi_3 = \frac{2}{3}x^3 - \frac{1}{2}x^4$ and taking $w_i = \psi_i$, $i = 1, 2, 3$, we have

$$\int_0^1 \left(2x - x^2\right)\left(-\frac{d^2}{dx^2}\left[x + c_1(2x - x^2) + c_2\left(x^2 - \frac{2}{3}x^3\right) + c_3\left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\right] - \beta\left[x + c_1(2x - x^2) + c_2\left(x^2 - \frac{2}{3}x^3\right) + c_3\left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\right] + x^2\right)dx = 0$$

$$\int_0^1 \left(x^2 - \frac{2}{3}x^3\right)\left(-\frac{d^2}{dx^2}\left[x + c_1(2x - x^2) + c_2\left(x^2 - \frac{2}{3}x^3\right) + c_3\left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\right] - \beta\left[x + c_1(2x - x^2) + c_2\left(x^2 - \frac{2}{3}x^3\right) + c_3\left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\right] + x^2\right)dx = 0$$

$$\int_0^1 \left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\left(-\frac{d^2}{dx^2}\left[x + c_1(2x - x^2) + c_2\left(x^2 - \frac{2}{3}x^3\right) + c_3\left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\right] - \beta\left[x + c_1(2x - x^2) + c_2\left(x^2 - \frac{2}{3}x^3\right) + c_3\left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\right] + x^2\right)dx = 0$$

Generating three necessary and sufficient random algebraic equations to determine the random coefficients c_1, c_2 and c_3 ,

$$U_3(x) = \psi_0 + c_1\psi_1(x) + c_2\psi_2(x) + c_3\psi_3(x)$$

$$= x - \frac{\left(231\beta^3 - 16742\beta^2 + 342384\beta - 225792\right)(2x - x^2)}{16\left(22\beta^3 - 2079\beta^2 + 39312\beta - 84672\right)}$$

$$+ \frac{21\left(3\beta^3 + 36\beta^2 - 1988\beta + 1344\right)(x^2 - \frac{2}{3}x^3)}{2\left(22\beta^3 - 2079\beta^2 + 39312\beta - 84672\right)}$$

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$$-\frac{21\left(19\beta^3-752\beta^2+1668\beta-2688\right)\left(\frac{2}{3}x^3-\frac{1}{2}x^4\right)}{4\left(22\beta^3-2079\beta^2+39312\beta-84672\right)}$$

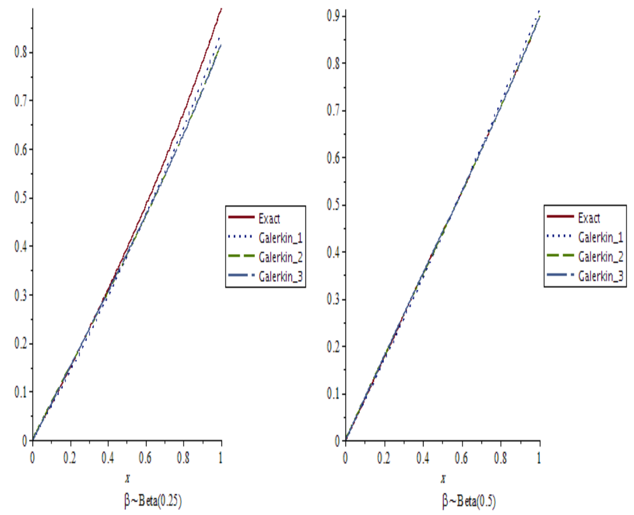

Fig 1: Random approximation and Exact solution with β ~ Beta distribution

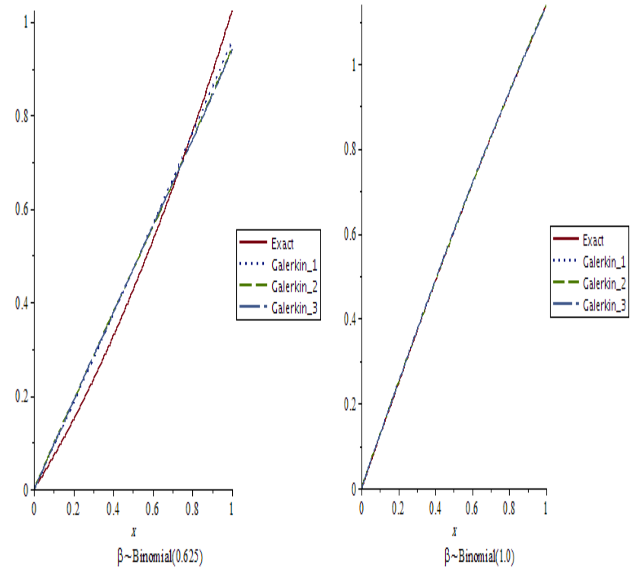

Fig 2: Random approximation and exact solution with β ~ Binomial distribution

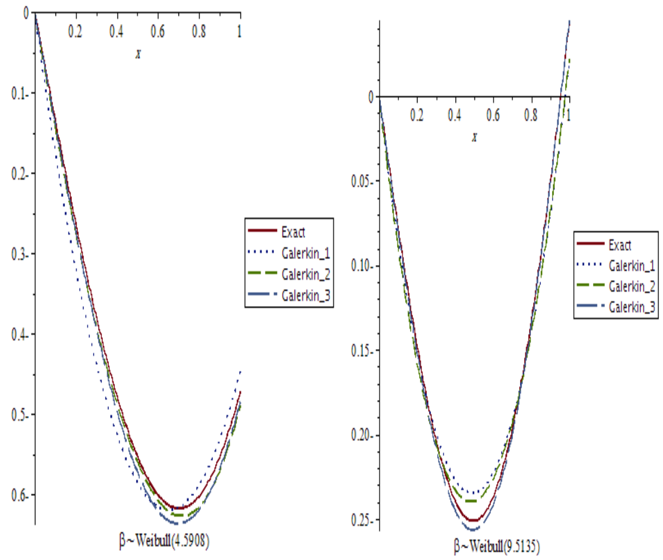

Fig 3: Random approximation and exact solution with β ~ Weibull distribution

Numerical Solution by Using theRandom Petrov-Galerkin Method.

For $n=1$, Choosing $\psi_0 = x$, $\psi_1 = 2x - x^2$ and taking w_i to be different from ψ_i , $w_1 = x$

$$\int_0^1 (x)\left(-\frac{d^2}{dx^2}\left[x+c_1\left(2x-x^2\right)\right]-\beta\left[x+c_1\left(2x-x^2\right)\right]+x^2\right)dx=0$$

$$\int_0^1 (x)\left(2c_1-\beta\left(x+c_1\left(2x-x^2\right)\right)+x^2\right)dx=0$$

$$\frac{1}{4}c_1\beta-\frac{1}{3}\beta(2c_1+1)+c_1+\frac{1}{4}=0$$

$$c_1=\frac{4\beta-3}{5\beta-12}$$

$$U_1(x)=\psi_0+c_1\psi_1(x)=x-\frac{4\beta-3}{5\beta-12}(2x-x^2)$$

For $n=2$, choosing $\psi_0 = x$, $\psi_1 = 2x - x^2$ and $\psi_2 = x^2 - \frac{2}{3}x^3$ and taking $w_1 = x$, $w_2 = x^2$, we have

$$\int_0^1 (x)\left(-\frac{d^2}{dx^2}\left[x+c_1(2x-x^2)+c_2\left(x^2-\frac{2}{3}x^3\right)\right]-\beta\left[x+c_1(2x-x^2)+c_2\left(x^2-\frac{2}{3}x^3\right)\right]+x^2\right)dx=0$$

$$\int_0^1 (x^2)\left(-\frac{d^2}{dx^2}\left[x+c_1(2x-x^2)+c_2\left(x^2-\frac{2}{3}x^3\right)\right]-\beta\left[x+c_1(2x-x^2)+c_2\left(x^2-\frac{2}{3}x^3\right)\right]+x^2\right)dx=0$$

Generating two necessary and sufficient random algebraic equations to determine the random coefficients c_1 and c_2, we get

$$\frac{2}{15}c_2\beta-\frac{1}{4}\beta(c_2-c_1)+\frac{1}{3}c_2-\frac{1}{4}\beta(2c_1+1)+c_1+\frac{1}{4}=0$$

$$\frac{1}{9}c_2\beta-\frac{1}{5}\beta(c_2-c_1)+\frac{1}{3}c_2-\frac{1}{4}\beta(2c_1+1)+\frac{2}{3}c_1+\frac{1}{5}=0$$

$$U_2(x)=\psi_0+c_1\psi_1(x)+c_2\psi_2(x)$$

$$=x-\frac{\left(5\beta^2-288\beta+180\right)(2x-x^2)}{2\left(11\beta^2-270\beta+600\right)}-\frac{15\left(3\beta^2-26\beta+24\right)(x^2-\frac{2}{3}x^3)}{2\left(11\beta^2-270\beta+600\right)}$$

For $n=3$, choosing $\psi_0=x$, $\psi_1=2x-x^2$, $\psi_2=x^2-\frac{2}{3}x^3$ and $\psi_3=\frac{2}{3}x^3-\frac{1}{2}x^4$ andtaking $w_1=x$, $w_2=x^2$, $w_3=x^3$, we have

$$\int_0^1 (x)\left(-\frac{d^2}{dx^2}\left[x+c_1(2x-x^2)+c_2\left(x^2-\frac{2}{3}x^3\right)+c_3\left(\frac{2}{3}x^3-\frac{1}{2}x^4\right)\right]-\beta\left[x+c_1(2x-x^2)+c_2\left(x^2-\frac{2}{3}x^3\right)+c_3\left(\frac{2}{3}x^3-\frac{1}{2}x^4\right)\right]+x^2\right)dx=0$$

$$\int_0^1 (x^2)\left(-\frac{d^2}{dx^2}\left[x+c_1(2x-x^2)+c_2\left(x^2-\frac{2}{3}x^3\right)+c_3\left(\frac{2}{3}x^3-\frac{1}{2}x^4\right)\right]-\beta\left[x+c_1(2x-x^2)+c_2\left(x^2-\frac{2}{3}x^3\right)+c_3\left(\frac{2}{3}x^3-\frac{1}{2}x^4\right)\right]+x^2\right)dx=0$$

$$\int_0^1 (x^2)\left(-\frac{d^2}{dx^2}\left[x+c_1(2x-x^2)+c_2\left(x^2-\frac{2}{3}x^3\right)+c_3\left(\frac{2}{3}x^3-\frac{1}{2}x^4\right)\right]-\beta\left[x+c_1(2x-x^2)+c_2\left(x^2-\frac{2}{3}x^3\right)+c_3\left(\frac{2}{3}x^3-\frac{1}{2}x^4\right)\right]+x^2\right)dx=0$$

Generating three necessary and sufficient random algebraic equations to determine the random coefficients

$$\frac{1}{12}c_3\beta-\frac{1}{5}\beta(-\frac{2}{3}c_2+\frac{2}{3}c_3)+\frac{1}{6}c_3-\frac{1}{4}\beta(c_2-c_1)+\frac{1}{3}c_2-\frac{1}{3}\beta(2c_1+1)+c_1+\frac{1}{4}=0$$

$$\frac{1}{14}c_3\beta-\frac{1}{6}\beta(-\frac{2}{3}c_2+\frac{2}{3}c_3)+\frac{1}{5}c_3-\frac{1}{5}\beta(c_2-c_1)+\frac{1}{3}c_2-\frac{1}{4}\beta(2c_1+1)+\frac{2}{3}c_1+\frac{1}{5}=0$$

$$\frac{1}{16}c_3\beta-\frac{1}{7}\beta(-\frac{2}{3}c_2+\frac{2}{3}c_3)+\frac{1}{5}c_3-\frac{1}{6}\beta(c_2-c_1)+\frac{3}{10}c_2-\frac{1}{5}\beta(2c_1+1)+\frac{1}{2}c_1+\frac{1}{6}=0$$

$$U_3(x)=\psi_0+c_1\psi_1(x)+c_2\psi_2(x)+c_3\psi_3(x)$$

$$=x-\frac{\left(29\beta^3-1784\beta^2+35700\beta-23520\right)(2x-x^2)}{2\left(19\beta^3-1734\beta^2+32760\beta-70560\right)}$$
$$+\frac{7\left(13\beta^3+34\beta^2-4920\beta+3360\right)(x^2-\frac{2}{3}x^3)}{2\left(19\beta^3-1734\beta^2+32760\beta-70560\right)}$$
$$-\frac{56\left(2\beta^3-64\beta^2+135\beta-210\right)(\frac{2}{3}x^3-\frac{1}{2}x^4)}{19\beta^3-1734\beta^2+32760\beta-70560}$$

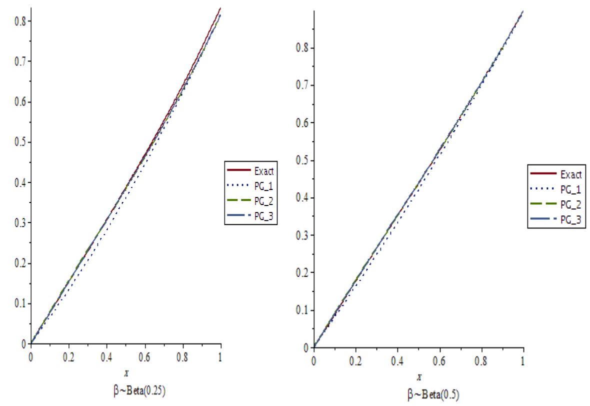

Fig 4: Random approximation and exact solution with β ~ Beta distribution

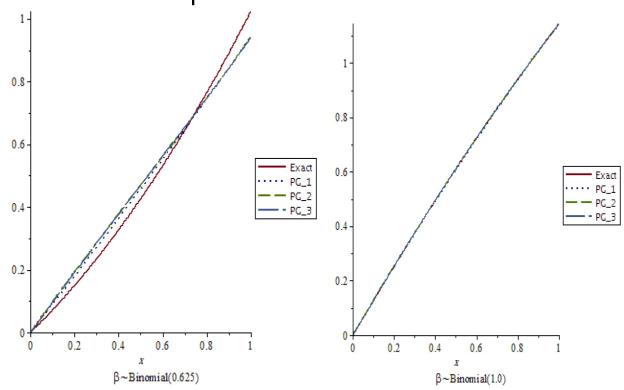

Fig 5: Random approximation and Exact solution with β ~ Binomial distribution

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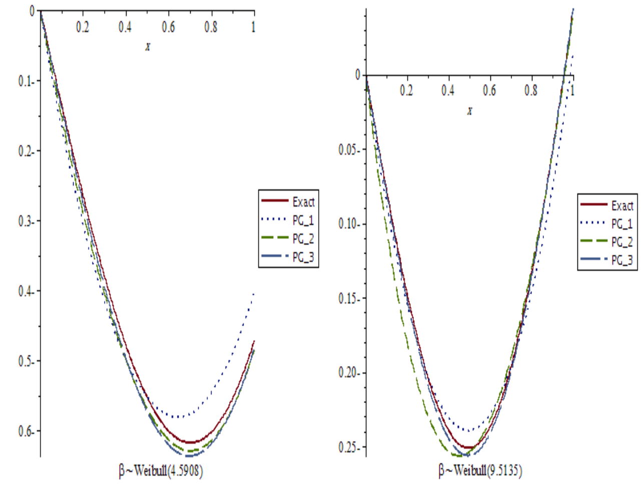

Fig 6: Random approximation and Exact solution with β ~ Weibull distribution

Numerical Solution using the random Least-Squares Method

For $n=1$, Choosing $\psi_0 = x$ and $\psi_1 = 2x - x^2$ and taking $w_i = \frac{\partial R}{\partial c_i}$ (i.e., $w_1 = \frac{\partial R}{\partial c_1}$)

$$w1 = \frac{\partial}{\partial c1}\left(-\frac{d^2}{dx^2}\left(x + c1\left(2x - x^2\right)\right) - \beta\left(x + c1\left(2x - x^2\right)\right) + x^2\right) = 2 - \beta\left(2x - x^2\right)$$

$$\int_0^1\left[2 - \beta\left(2x - x^2\right)\right]\left[-\frac{d^2}{dx^2}\left(x + c1\left(2x - x^2\right)\right) - \beta\left(x + c1\left(2x - x^2\right)\right) + x^2\right]dx = 0$$

$$\int_0^1\left(2 - \beta\left(2x - x^2\right)\right)\left[2c1 - \beta\left(x + c1\left(2x - x^2\right)\right) + x^2\right]dx = 0$$

$$-\frac{3}{10}\beta(\beta c1 + 1) + \frac{5}{12}\beta^2(2c1 + 1) - \frac{2}{\%3}\beta c1 - \beta(2c1 + 1) + 4c1 + \frac{2}{3} = 0$$

$$c_1 = -\frac{1}{16}\frac{25\beta^2 - 78\beta + 40}{2\beta^2 - 10\beta + 15}$$

$$U_1(x) = \psi_0 + c_1\psi_1(x) = x - \frac{25\beta^2 - 78\beta + 40}{16\left(2\beta^2 - 10\beta + 15\right)}(2x - x^2)$$

For $n=2$, choosing $\psi_0 = x$, $\psi_1 = 2x - x^2$ and $\psi_2 = x^2 - \frac{2}{3}x^3$ and taking $w_i = \frac{\partial R}{\partial c_i}$, $i = 1, 2$, we have

$$\begin{aligned} w_1 = \frac{\partial}{\partial c_1}\Bigg(&-\frac{d^2}{dx^2}\left[x + c_1(2x - x^2) + c_2\left(x^2 - \frac{2}{3}x^3\right)\right] \\ &- \beta\left[x + c_1(2x - x^2) \right. \\ &\left. + c_2\left(x^2 - \frac{2}{3}x^3\right)\right] + x^2\Bigg) \\ &= 2 - \beta(2x - x^2) \end{aligned}$$

$$\begin{aligned} \int_0^1 (2 - \beta(2x - x^2))\Bigg(-\frac{d^2}{dx^2}&\left[x + c_1(2x - x^2) \right. \\ &\left. + c_2\left(x^2 - \frac{2}{3}x^3\right)\right] \\ &- \beta\left[x + c_1(2x - x^2) \right. \\ &\left. + c_2\left(x^2 - \frac{2}{3}x^3\right)\right] + x^2\Bigg)dx = 0 \end{aligned}$$

$$\begin{aligned} w_2 = \frac{\partial}{\partial c_2}\Bigg(&-\frac{d^2}{dx^2}\left[x + c_1(2x - x^2) + c_2\left(x^2 - \frac{2}{3}x^3\right)\right] \\ &- \beta\left[x + c_1(2x - x^2) \right. \\ &\left. + c_2\left(x^2 - \frac{2}{3}x^3\right)\right] + x^2\Bigg) \\ &= 4x - 2 - \beta\left(x^2 - \frac{2}{3}x^3\right) \end{aligned}$$

$$\begin{aligned} \int_0^1 \left(4x - 2 - \beta\left(x^2 - \frac{2}{3}x^3\right)\right)\Bigg(-\frac{d^2}{dx^2}&\left[x + c_1(2x - x^2) \right. \\ &\left. + c_2\left(x^2 - \frac{2}{3}x^3\right)\right] \\ &- \beta\left[x + c_1(2x - x^2) \right. \\ &\left. + c_2\left(x^2 - \frac{2}{3}x^3\right)\right] + x^2\Bigg)dx = 0 \end{aligned}$$

Generating two necessary and sufficient random algebraic equations to determine the random coefficients c_1 and c_2 ,

$$\frac{13}{90}c_2\beta^2 + \frac{8}{15}c_1\beta^2 - \frac{13}{10}\beta - \frac{2}{3}c_2\beta + \frac{5}{12}\beta^2 - \frac{8}{3}c_1\beta + 4c_1 + \frac{2}{3} = 0$$

$$\frac{13}{315}c_2\beta^2 + \frac{13}{90}c_1\beta^2 - \frac{19}{45}\beta - \frac{4}{15}c_2\beta + \frac{7}{60}\beta^2 - \frac{2}{3}c_1\beta + \frac{4}{3}c_1 + \frac{1}{3} = 0$$

$$\begin{aligned} U_2(x) &= \psi_0 + c_1\psi_1(x) + c_2\psi_2(x) \\ &= x - \frac{\left(39\beta^4 - 2948\beta^3 + 68052\beta^2 - 191520\beta + 100800\right)(2x - x^2)}{2\left(65\beta^4 - 3384\beta^3 + 64800\beta^2 - 262080\beta + 302400\right)} \\ &\quad - \frac{21\left(11\beta^4 - 382\beta^3 + 4360\beta^2 - 11520\beta + 7200\right)(x^2 - \frac{2}{3}x^3)}{2\left(65\beta^4 - 3384\beta^3 + 64800\beta^2 - 262080\beta + 302400\right)} \end{aligned}$$

For $n=3$, choosing $\psi_0 = x$, $\psi_1 = 2x - x^2$, $\psi_2 = x^2 - \frac{2}{3}x^3$ and $\psi_3 = \frac{2}{3}x^3 - \frac{1}{2}x^4$ and taking $w_i = \frac{\partial R}{\partial c_i}$, $i = 1, 2, 3$ we have

$$R(x, c_i) = 2c_1 - c_2(-4x + 2) - c_3(-6x^2 + 4x)$$
$$-\beta\left(x + c_1\left(2x - x^2\right) + c_2\left(x^2 - \frac{2}{3}x^3\right) + c_3\left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\right) + x^2$$

$$w_1 = 2 - \beta\left(2x - x^2\right)$$

$$w_2 = 4x - 2 - \beta\left(x^2 - \%\frac{2}{3}x^3\right)$$

$$w_3 = 6x^2 - 4x - \beta\left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)$$

$$\int_0^1 w_i R(x, c_i)dx = 0,\ i = 1,2,3 \quad (12)$$

Generates three necessary and sufficient random algebraic equations to determine the random coefficients

$$\frac{19}{315}c_3\beta^2+\frac{13}{90}c_2\beta^2+\frac{8}{15}c_1\beta^2-\frac{4}{15}c_3\beta-\frac{2}{3}c_2\beta-\frac{8}{3}c_1\beta+\frac{5}{12}\beta^2-\frac{13}{10}\beta+4c_1+\frac{2}{3}=0$$

$$\frac{1}{56}c_3\beta^2+\frac{13}{315}c_2\beta^2+\frac{13}{90}c_1\beta^2-\frac{2}{15}c_3\beta-\frac{4}{15}c_2\beta-\frac{2}{3}c_1\beta+\frac{7}{60}\beta^2-\frac{19}{45}\beta+\frac{2}{3}c_3+\frac{4}{3}c_2+\frac{1}{3}=0$$

$$\frac{1}{126}c_3\beta^2+\frac{1}{56}c_2\beta^2+\frac{19}{315}c_1\beta^2-\frac{8}{105}c_3\beta-\frac{2}{15}c_2\beta-\frac{4}{15}c_1\beta+\frac{1}{20}\beta^2-\frac{13}{63}\beta+\frac{8}{15}c_3+\frac{2}{3}c_2+\frac{1}{5}=0$$

So, we can find the value of the coefficients and have U_3.

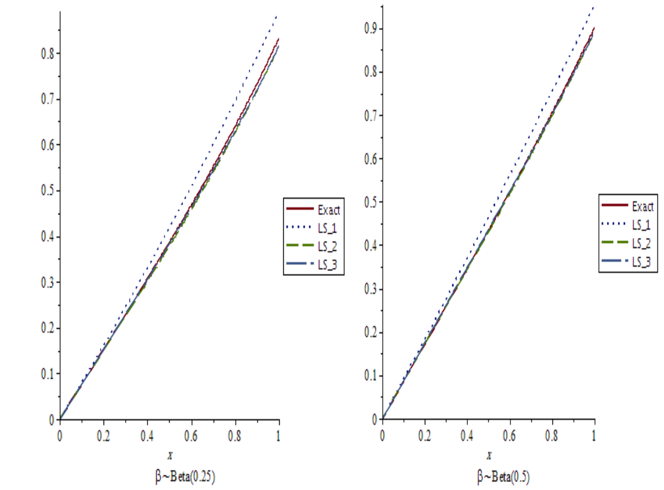

Fig 7: Random approximation and exact solution with β ~ Beta distribution

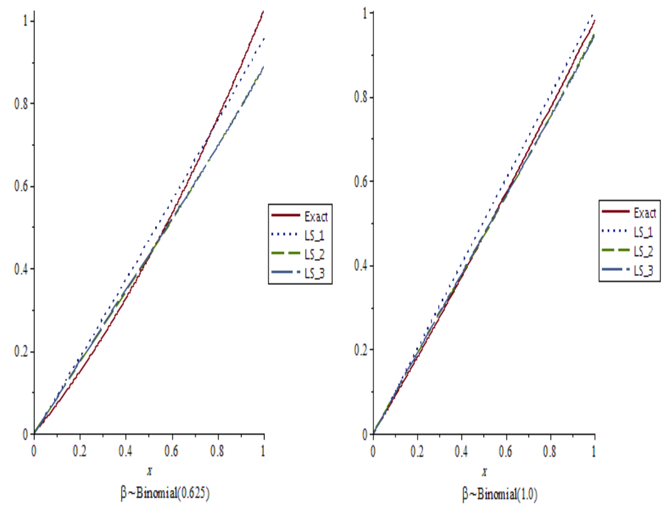

Fig 8: Random approximation and exact solution with β ~ Binomial distribution

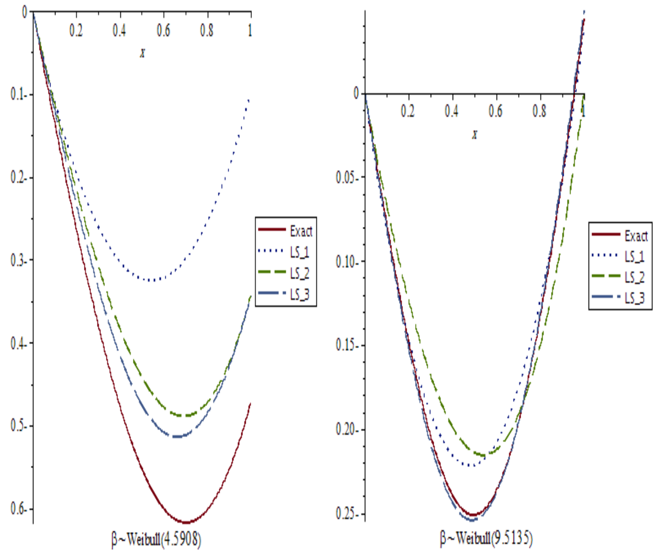

Fig 9: Random approximation and exact solution with β ~ Weibull distribution

Numerical Solution using the random Collocation Method

For $n = 1$, choosing $\psi_0 = x$ and $\psi_1 = 2x - x^2$, and choosing $x = \frac{1}{3}$ as a collocation point, we have:

$$R = -\frac{d^2}{dx^2}\left(x + c_1\left(2x - x^2\right)\right) - \beta\left(x + c_1\left(2x - x^2\right)\right) + x^2$$

Evaluating the random residual at this collocation point and setting it equal to zero,

$$R(\frac{1}{3}) = 2c_1 - \beta\left(\frac{1}{3} + \frac{5}{9}c_1\right) = 0$$

$$c_1 = -\frac{3\beta - 1}{5\beta - 18}$$

$$U_1(x) = \psi_0 + c_1\psi_1(x) = x - \frac{3\beta - 1}{5\beta - 18}(2x - x^2)$$

For $n = 2$, choosing $\psi_0 = x$, $\psi_1 = 2x - x^2$ and $\psi_2 = x^2 - \%\frac{2}{3}x^3$, and choosing $x = \frac{1}{3}$ and $\frac{1}{2}$ as thecollocation points, we have:

$$R = -\frac{d^2}{dx^2}\left(x + c_1\left(2x - x^2\right) + c_2\left(x^2 - \frac{2}{3}x^3\right)\right)$$
$$-\beta\left(x + c_1\left(2x - x^2\right) + c_2\left(x^2 - \frac{2}{3}x^3\right)\right) + x^2 = 0$$

Evaluating the random residuals at these collocation points and set them equal to zero, we will get the two necessary and sufficient random algebraic equations todetermine the coefficients

$$R(\frac{1}{3}) = 2c_1 - \frac{2}{3}c_2 - \frac{1}{3}\beta - \frac{5}{9}c_1\beta - \frac{7}{81}c_2\beta + \frac{1}{9} = 0$$

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$$R(\frac{1}{2}) = 2c_1 - \frac{1}{2}\beta - \frac{3}{4}c_1\beta - \frac{1}{6}c_2\beta + \frac{1}{4} = 0$$

$$U_2(x) = \psi_0 + c_1\psi_1(x) + c_2\psi_2(x)$$

$$= x - \frac{\left(4\beta^2 - 107\beta + 54\right)(2x - x^2)}{9\beta^2 - 214\beta + 432} - \frac{9\left(\beta^2 - 14\beta + 10\right)(x^2 - \frac{2}{3}x^3)}{9\beta^2 - 214\beta + 432}$$

For $n = 3$, choosing $\psi_0 = x$, $\psi_1 = 2x - x^2$, $\psi_2 = x^2 - \frac{2}{3}x^3$ and $\psi_3 = \frac{2}{3}x^3 - \frac{1}{2}x^4$, andchoosing $x = \frac{1}{3}, \frac{1}{2}$ and $\frac{2}{3}$ as the collocationpoints, we have

$$R = 2c_1 - c_2(-4x + 2) - c_3(-6x^2 + 4x)$$
$$-\beta\left(x + c_1\left(2x - x^2\right) + c_2\left(x^2 - \frac{2}{3}x^3\right) + c_3\left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\right) + x^2$$

$$R(\frac{1}{3}) = 2c_1 - \frac{2}{3}c_2 - \frac{2}{3}c_3 - \frac{1}{3}\beta - \frac{5}{9}c_1\beta - \frac{7}{81}c_2\beta - \frac{1}{54}c_3\beta + \frac{1}{9} = 0$$

$$R(\frac{1}{2}) = 2c_1 - \frac{1}{2}c_3 - \frac{1}{2}\beta - \frac{3}{4}c_1\beta - \frac{1}{6}c_2\beta - \frac{5}{96}c_3\beta + \frac{1}{4} = 0$$

$$R(\frac{2}{3}) = 2c_1 + \frac{2}{3}c_2 - \frac{2}{3}\beta - \frac{8}{9}c_1\beta - \frac{20}{81}c_2\beta - \frac{8}{81}c_3\beta + \frac{4}{9} = 0$$

$$U_3(x) = \psi_0 + c_1\psi_1(x) + c_2\psi_2(x) + c_3\psi_3(x)$$

$$= x - \frac{\left(17\beta^3 - 895\beta^2 + 11979\beta - 7776\right)(2x - x^2)}{2\left(15\beta^3 - 844\beta^2 + 11502\beta - 23328\right)}$$

$$+ \frac{9\left(\beta^3 + 49\beta^2 - 1295\beta + 864\right)(x^2 - \frac{2}{3}x^3)}{2\left(15\beta^3 - 844\beta^2 + 11502\beta - 23328\right)}$$

$$- \frac{36\left(\beta^3 - 29\beta^2 + 67\beta - 108\right)(\frac{2}{3}x^3 - \frac{1}{2}x^4)}{15\left(15\beta^3 - 844\beta^2 + 11502\beta - 23328\right)}$$

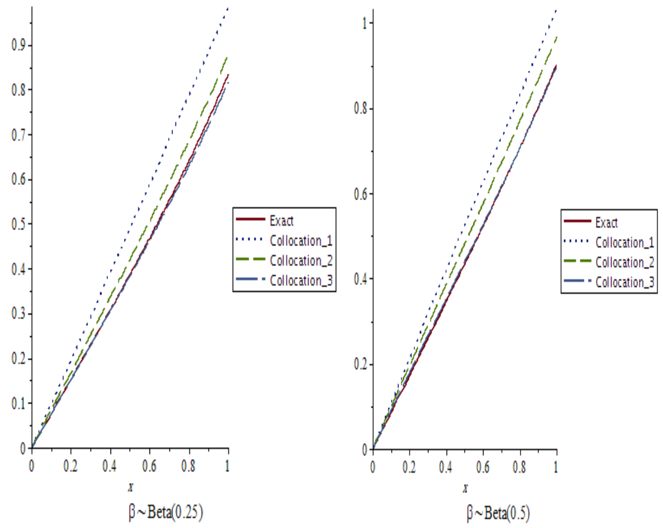

Fig 10: Random approximation and exact solution with β ~ Beta distribution

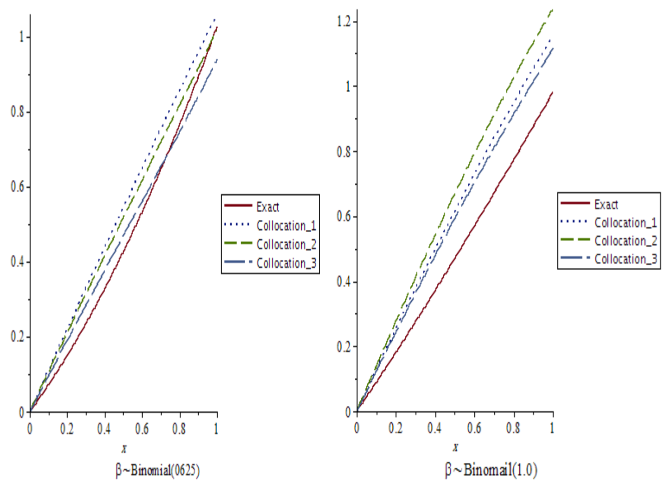

Fig 11: Random approximation and exact solution with β ~ Binomial distribution

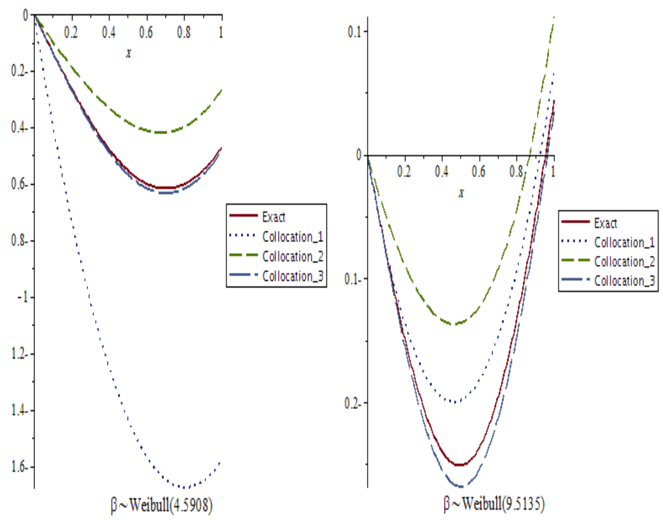

Fig 12: Random approximation and exact solution with β ~ Weibull distribution

Example 2. The following random one dimensional heat problem:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} & \text{in } R \times T \times \Omega \\ u(x,0) = \beta \sin(\pi x) & \text{in } R \times \Omega \quad (13) \\ u(0,t) = 0,\; u(L,t) = 0 & \text{in } \partial R \times T \times \Omega \end{cases}$$

where β is a second bounded order random variable.

The exact solution:

$$u(x,t) = e^{-\pi^2 t} \beta \sin(\pi x) \quad (14)$$

Numerical Solution using the random Galerkin Method

Choosing $\psi_0 = 0$ so that satisfies the actualboundary conditions $\psi_0(0) = 0$ and $\psi_0(1) = 0$, taking $w_i = \psi_i$ (Galerkin)where $\psi_j = x^j(1-x)$ so that satisfies the homogeneous form of boundary conditions.

For $n = 1$, $\psi_1 = w_1 = x(1-x)$ (Galerkin Approach) and assuming L=1cm,we have:

$$A_{ij} = \int_0^1 x^{i+j}(1-x)^2 dx$$

$$F_{ij} = \int_0^1 x^i (1-x) \frac{\partial^2}{\partial x^2}\left[x^j (1-x)\right] dx$$

By (7), we get

$$\frac{1}{30}\frac{\partial}{\partial t} c_1(t) = -\frac{1}{3} c_1(t) \quad (15)$$

(15) is an ordinary differential equation, for solving this one we needthe initial condition of the random coefficients vector at $t = 0$.

$$\bar{c}(0) = A^{-1}\int_0^1 \beta \sin(\pi x)\psi_i(x) dx$$

The initial condition of the previous random ordinary differential equation is

$$\bar{c}(0) = 30\int_0^1 \beta \sin(\pi x)(x - x^2) dx = 3.8655\beta$$

The solution subject to the initial condition is:

$$u_1(x,t) = 3.8655\beta e^{-10t}\left(x - x^2\right)$$

For $n = 2$:

$$A = \begin{bmatrix} \frac{1}{30} & \frac{1}{60} \\ \frac{1}{60} & \frac{1}{105} \end{bmatrix}, F = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{2}{15} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{30} & \frac{1}{60} \\ \frac{1}{60} & \frac{1}{105} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t} c_1(t) \\ \frac{\partial}{\partial t} c_2(t) \end{bmatrix} = -\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{15} \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$$

$$\bar{c}(0) = A^{-1}\int_0^1 \beta \sin(\pi x)\psi_i(x) dx$$

$$= \begin{bmatrix} \frac{160}{3} & -\frac{140}{3} \\ -\frac{140}{3} & \frac{280}{3} \end{bmatrix} \begin{bmatrix} 0.1288 \\ 0.0644 \end{bmatrix} \beta = \begin{bmatrix} 3.8655 \\ 0 \end{bmatrix} \beta$$

Solving the system of an ordinary random differential equations subject to the random initial condition of coefficients vector, one gets

$$u_2(x,t) = 3.8779\beta e^{-10t}\left(x - x^2\right)$$

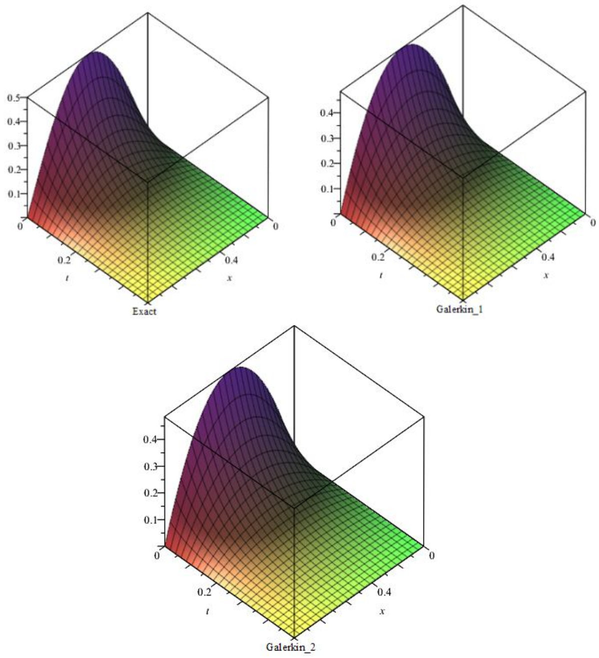

Fig 13: Random approximation and exact solution with β ~ N(1.0,2.0)

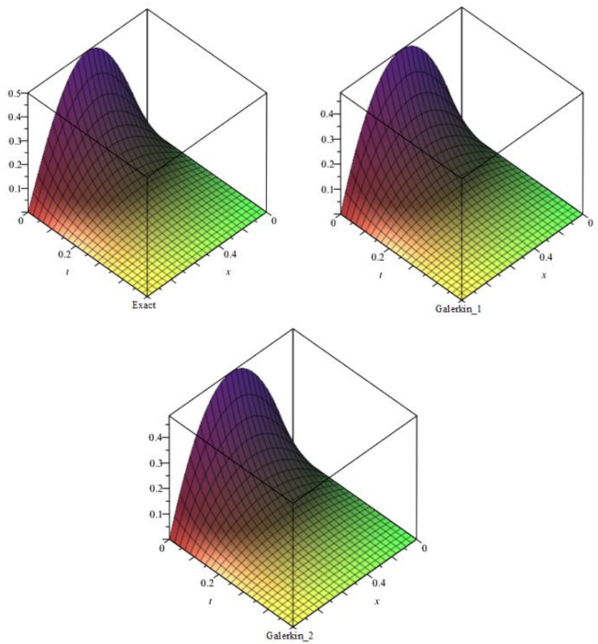

Fig 14: Random approximation and Exact solution with β ~ Beta (1.0,1.0)

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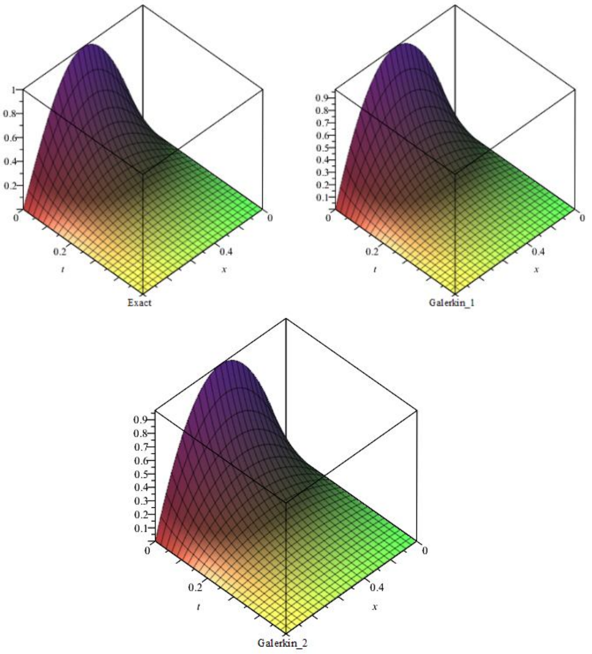


Fig 15: Random approximation and exact solution with β ~ Poisson (0.5)

Numerical Solution using the random Petrov-Galerkin Method

Choosing $\psi_0 = 0$ so that satisfies the actual boundary conditions $\psi_0(0) = 0$ and $\psi_0(1) = 0$, taking $w_i \neq \psi_i$. Choosing $\psi_j = x^j$ to be different from approximation functions and $\psi_j = x^j(1-x)$

$$A_{ij} = \int_0^1 x^{i+j}(1-x)dx$$

$$F_{ij} = \int_0^1 x^i \frac{\partial^2}{\partial x^2}\left[x^j(1-x)\right]dx$$

For $n = 1$, $\psi_1 = x(1-x)$, $w_1 = x$ and assuming L=1 cm,we have

$$\frac{1}{12}\frac{\partial}{\partial t}c_1(t) = -c_1(t) \quad (7)$$

(16) is an ordinary differential equation, for solving this one we needthe initial condition of coefficients vector at $t = 0$.

$$\bar{c}(0) = A^{-1}\int_0^1 \beta \sin(\pi x)\psi_i(x)dx$$

The initial condition of the previous random ordinary differential equationis :

$$\bar{c}(0) = 12\int_0^1 \beta \sin(\pi x)(x)dx = 3.8181\beta$$

The solution subject to the initial condition is:

$$u_1(x,t) = 3.8181\beta e^{-12t}\left(x - x^2\right)$$

For $n = 2$:

$$A = \begin{bmatrix} \frac{1}{12} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{30} \end{bmatrix}, F = \begin{bmatrix} -1 & -1 \\ -\frac{2}{3} & -\frac{5}{6} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{12} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{30} \end{bmatrix}\begin{bmatrix} \frac{\partial}{\partial t}c_1(t) \\ \frac{\partial}{\partial t}c_2(t) \end{bmatrix} = -\begin{bmatrix} 1 & 1 \\ \frac{2}{3} & \frac{5}{6} \end{bmatrix}\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$$

$$\bar{c}(0) = A^{-1}\int_0^1 \beta \sin(\pi x)\psi_i(x)dx$$

$$= \begin{bmatrix} 120 & -180 \\ -180 & 300 \end{bmatrix}\begin{bmatrix} 0.3182 \\ 0.1893\% \end{bmatrix}\beta = \begin{bmatrix} 4.0980 \\ -0.4670 \end{bmatrix}\beta$$

Solving the system of ordinary random differential equations subject to the random initial condition of coefficients vector, one gets,

$$u_2(x,t) = \left(-0.5394\beta e^{-60t} + 4.6374\beta e^{-10t}\right)\left(x - x^2\right)$$
$$+\left(0.5394\beta e^{-60t} - 1.5458\beta e^{-10t}\right)\left(x^2 - x^3\right)$$

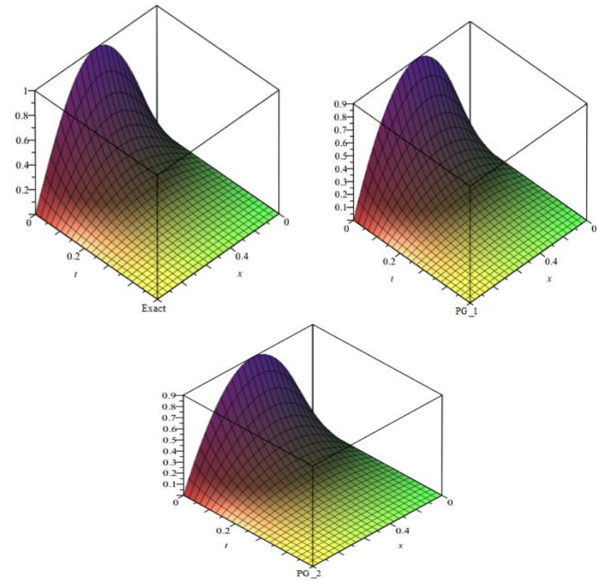
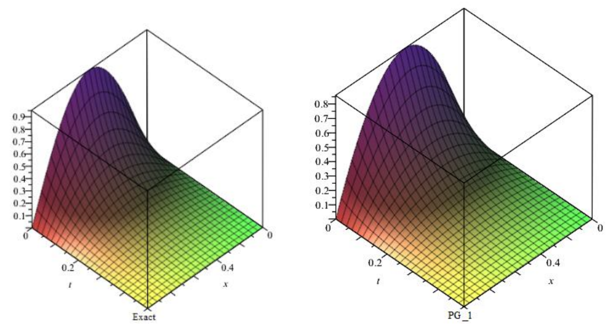


Fig 16: Random approximation and exact solution with β ~ N(1.0,2.0)



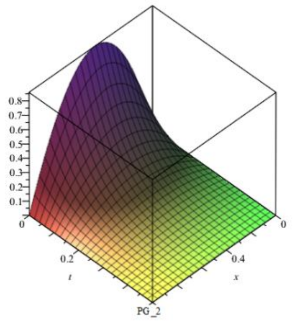


Fig 15: Random approximation and exact solution with $\beta \sim$ Poisson (0.5)

VI. CONCLUSION

We have discussed in this work that if we want to use the variational methods for solving random models, the randomness input must be bounded and this is shown by some numerical case studies.

CONFLICT OF INTEREST STATEMENT

The authors declare that there is no conflict of interests regarding the publication of this article.

REFERENCES

[1] M. A. SOHALY, Mean square convergent three and five points finite difference scheme for stochastic parabolic partial differential equations, Electronic Journal of Mathematical Analysis and Applications 2 (14) (2014) 164–171.
[2] W. Ames, Numerical Methods for Partial Differential Equations, Computer Science and Scientific Computing, Elsevier Science, 2014.
[3] M. A. El-Tawil, M. A. Sohaly, Mean square numerical methods for initial value random differential equations,Open Journal of Discrete Mathematics 1 (2) (2011) 66–84.
[4] J. Cortes, L. Jodar, L. Villafuerte, R. Villanueva, Computing mean square approximations of random diffusionmodels with source term, Mathematics and Computers in Simulation 76 (2007) 44–48.
[5] M. Yassen, M. Sohaly, I. Elbaz, Random crank-nicolson scheme for random heat equation in mean square sense,American Journal of Computational Mathematics 6 (2016) 66–73.
[6] A. Mitchell, D. Griffith, The finite difference method in partial differential equations.
[7] J.N. Reddy, An Introduction to the Finite Element Method, Third Edition, McGraw-Hill, New York, 2006.
[8] B. Szabo, I. Babuˇska, Finite Element Analysis, A Wiley-Interscience publication, 1991.
[9] J. T. Oden, J. N. Reddy, Variational methods in theoretical mechanics, Springer Science & Business Media, 2012.
[10] J.N. Reddy, D. K. Gartling, the finite element method in heat transfer and fluid dynamics, CRC press, 2010.
[11] O. Zienkiewicz, R. Taylor, The Finite Element Method: Solid mechanics, Butterworth-Heinemann, 2000.
[12] R. Wait, A. Mitchell, The Finite Element Method in Partial Differential Equations, John Willey and Sons, 1978.
[13] C. A. J. Fletcher, Computational Galerkin Methods, Springer Berlin Heidelberg, 1984.
[14] N. Young, An Introduction to Hilbert Space, Cambridge mathematical textbooks, Cambridge University Press,1988.
[15] T. T. Soong, Random Differential Equations in Science and Engineering, Academic Press, New York, 1973.
[16] K. Kuratowski, Topology, Elsevier Science, 2014.
[17] L. Debnath, F. A. Shah, Hilbert Spaces and Orthonormal Systems, Birkh¨auser Boston, Boston, MA, 2015, pp.29–127.
[18] R. Burden, J. Faires, Numerical Analysis, Cengage Learning, 2010.
[19] Back Matter, ACADEMIC PRESS New York and London, 1972. doi:10.1137/1.9781611973242.bm.
[20] J. Oden, J. Reddy, An Introduction to the Mathematical Theory of Finite Elements, Dover Books on Engineering,Dover Publications, 2012.
[21] S. Mikhlin, The numerical performance of variational methods, Wolters-Noordho_ Series of Monographs andTextbooks on Pure and Applied Mathematics, Wolters-Noordho_ Publishing, 1971.

