Semi-Regular Group Divisible Designs
For Smaller Block Size

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Abstract: A group divisible (GD) design is said to be Singular (S) if \( r - \lambda_1 = 0 \); Semi regular (SR) if \( r - \lambda_1 > 0 \) and \( rk - v \lambda_2 = 0 \); Regular (R) if \( r - \lambda_1 > 0 \) and \( rk - v \lambda_2 > 0 \). In the paper, a new procedure of constructing SRGD design with \( k = m \) and \( \lambda_1 = 0 \), is proposed from a parent SRGD after reducing number of treatment and same number of blocks without disturbing its Semi-Regularity property. It privileges the experimenters to decrease the number of treatment without affecting the number of blocks. Such designs are useful in civil engineers, Agricultural experiments and others.

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I. INTRODUCTION

The methods of constructing resolvable group divisible designs are available in many literatures. Meitei [1] has constructed an increase of block-size of SRGD from the parent SRGD design by 1, making an increase of treatment numbers of the parent SRGD design, by \( n \). Dey and Nigam [10] have suggested a construction method of GD design with parameters \( (v = v's, b = b's, r = r', k = k', \lambda_1 = 0, \lambda_2 = s \lambda_2'; m = m', n = t') \), starting from another GD designs with parameters \( v^* = m^* n^*, b^*, r^*, k^*, \lambda_1^* = 0, \lambda_2^*; m^*, n^* = st \) \((s \geq 2, t \geq 2)\). If the starting design is SRGD i.e. \( r^* \lambda_1^* > 0 \) and \( r^* k^* - v^* \lambda_2^* = 0 \), then the resultant design remains as Semi-regular.

Many relations among the parameters of the design have been proposed by Bose and Connor [4] for the existence of Group Divisible (GD). Using combinatorial methods and the orthogonal arrays, many series of GD designs are presented in the literature of Bose, Shrikhande and Bhattacharya [5] and Raghava Rao [7]. Kageyama and Tsuji [9] have proposed \((i)\) a GD design is Semi-Regular iff \( k/m \) is an integer and every block contains exactly \( k/m \) treatment(s) from each group of the association scheme and \((ii)\) a GD design is singular iff \( k/n \) is integer and every block contains exactly \( k/n \) groups of the association scheme. Kageyama [11] has constructed \( \alpha \)-resolvable regular GD design starting from \( \alpha \)-resolvable Balanced Incomplete Block (BIB) design.

Orthogonal Array: A \( k \times n \) matrix \( B \) with entries from a set \( S \), containing \( s(\geq 2) \) symbols is said to be an orthogonal array with \( k \) constraints, \( N \) assemblies, \( s \) symbols and strength \( \ell \) if every \( t \times N \) sub-matrix of \( B \) contains each of the \( s^t \) \( t \)-plats of each symbol \( s \) equally often, say, \( \lambda \) times each where \( t \leq k \). The integer \( \lambda \) is called the index of the array. It is denoted by \( A(N, k, s, \ell) \). There are \( s^t \) possible \( t \)-plats of elements from \( S \) having \( s \) elements. Each possible \( s^t \) \( t \)-plat occurs \( \lambda \) times in every \( t \times N \) sub-matrix of \( B \). Thus \( N = \lambda s^t \).

By the definition, it is obvious that the existence of \( A(N, k, s, \ell) \) implies that of \( A(N, k, s, t_1) \) where \( t_1 < t \). However its index will change accordingly.

II. CONSTRUCTION

Let us consider a SRGD design, \( D \) with parameters \( v = ma, b = n^2 \lambda_2, r = n \lambda_2, k = m, \lambda_1 = 0, \lambda_2 = m, n \). Further, let \( n \left(i - 1\right) + j \) be the \( j^{th} \) treatment in the \( i^{th} \) group of the association scheme; \( i = 1, ..., m; j = 1, ..., n \). Since \( k \) is divisible by \( m \) \((i.e.k/m = 1)\) Kageyama and Tsuji [9], every block of \( D \) contains exactly one element from each group. Let us define \( m \) groups of the Group Divisible association scheme of \( D \), as given by \( G_i = \{ \left( i - 1\right)n + 0, \left( i - 1\right)n + 1, ..., (i - 1)n + n - 1 \} \) for \( i = 1, ..., m \). The block of SRGD design \( D \) is written as columns such that the element \( i^{th} \) group is placed in the \( i^{th} \) row. That is, the blocks structure is such that each of all the \( 1^{st} \) positions of the blocks are occupied by each of the
elements from the 1st group; that of the 2nd positions by that of the elements from the 2nd group, so on. Replacing the treatment \((i - 1)n + \theta\) of the \(i\)th group by \(\theta\), for \(\theta = 0, 1, \ldots, n - 1\), we get an orthogonal array, \(A[n^2\lambda_2, m, n, 2]\), due to Bose et al [5].

Deleting the last \(m - m_1\) rows from the orthogonal array \(A\) without disturbing the \(1^\text{st}\) \(m_1\) rows of \(A\), the parent orthogonal array gives \(A^*\), say, with \(n^2\lambda_2\) assemblies, \(m_1\) (< \(m\)) constraints, \(n\) symbols and strength 2 viz;

\[A^*=[n^2\lambda_2, m_1, n_1]\]  

(2.1)

Again, replacing the element \(\theta\) in the \(i\)th row of the array \(A^*\) with an element given by

\[(1-1)n + \theta; \theta = 0, 1, \ldots, n-1\]

i.e., \(\theta\) in the \(i\)th row of \(A^* \rightarrow (1-1)n + \theta \ldots \) (2.2)

The columns of the array \(A^*\), given by the relation (2.2), are the blocks of a resultant SRGD. Considering \(G_1, \ldots, G_{m_1}\) of the parent association scheme as the \(m_1\) groups of a resultant design, we can get the following theorem.

Theorem: The existence of a SRGD design with parameters \(v = mn\), \(b = n^2\lambda_2\), \(r = n\lambda_2\), \(k = m\), \(\lambda_1 = 0\), \(\lambda_2 = m\), \(m\), \(n\) implies that an SRGD design with parameters \(v' = m_1\) \(n^* = n^2\lambda_2\), \(r' = n\lambda_2\), \(k^* = m_1\) (< \(m\)), \(\lambda_1 = 0\), \(\lambda_2 = m\) is with \(m\) \(\lambda_2\) blocks of size \(m_1\) each (i.e., \(k^* = m_1\)). Further, each of \(m_1\) rows of the resultant orthogonal array \(A^*\) contains \(n\) symbols. Moreover, each element in each row of \(A^*\) produces distinct element as \(l = 1, \ldots, m_1\) (< \(m\)); \(\theta = 0, 1, \ldots, n - 1\) and the number of distinct elements of the resultant design is \(mm_1\) i.e., \(v^* = nm_1\). As the strength of the orthogonal array \(A^*\) is \(t\) (\(\geq 2\)), every \(2 \times n^2\lambda_2\) sub-matrix of the orthogonal array \(A^*\) contains each of \(n^2\) \(2\)-plats (viz; \((x_1, x_2), x_1, x_2 \in [0, 1, \ldots, n - 1])\) of the \(n\) symbols viz; \(0, 1, \ldots, n - 1\) as column, equally often, \(\lambda_2\) times [as \(N = \lambda n^2\)]. Then any symbols \(\theta\) (i.e., \(0, 1, \ldots, n - 1\)) every pair of \(\theta\) with each element of \(0, 1, \ldots, n - 1\) in column form appears \(\lambda_2\) times in \(A^*\). Then \(\theta\) appears \(n\lambda_2\) times. Any two treatments \((l - 1)n + \theta\) and \((l' - 1)n + \theta'; l \neq l'\); \(\psi, \phi = 0, 1, \ldots, n - 1\) of the resultant design are produced from the \(l\)th and \(l'\)th rows of \(A^*\) respectively. Consequently, \((l - 1)n + \theta + \psi\) and \((l' - 1)n + \phi + \theta\) belong to \(G_l\) and \(G_{l'}\) respectively by the relation (2.2) and are second associates to one another. Let us count in how many blocks of the resultant design they occur together. Any two treatments \((\psi, \phi)\) in \(A^*\) occurs \(\lambda_2\) times (in column form) i.e., \((\psi, \phi)^t\) in the sub-matrix \(2 \times n^2\lambda_2\) of \(A\) and accordingly of \(A^*\) also. As the index of an orthogonal array does not change when some rows of the concerned orthogonal array are removed. The \((l - 1)n + \psi\) and \((l' - 1)n + \phi\) occur together in \(\lambda_2\) blocks by the relation (2.2). So, \(\lambda_1^* = \lambda_2\). By the relation (2.2) the \(l\)th positions of any symbol of the resultant design are occupied by a single element of \(G_l (l = 0, 1, \ldots, m_1)\). Thus no block of the resultant design contains any two elements belonged to the same group of the association scheme. Therefore \(\lambda_1^* = 0\).

Using an isomorphic solution of SRGD design SR36, Clathworthy [6], an illustration is made below.

\[A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \end{bmatrix}\]

\[A^* = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \end{bmatrix}\]  

after deleting the last row of orthogonal array \(A\).

Rewriting the treatment \(\theta\) in the \(i\)th row of \(A^*\) by

\[\begin{align*} & (l - 1)2 + \theta; l = 1, 2, 3 \text{ and } \theta = 0, 1 \quad \text{ yields the 8 blocks and the 3 groups of the required SRGD design} \quad v^* = 6, \\ & b^* = 8, r^* = 4, k^* = 3, \lambda_1^* = 0, \lambda_2^* = 2, m^* = 3, n^* = 2. \end{align*}\]

As given
\[B_1 = \{0, 2, 4\}, \quad B_2 = \{1, 3, 5\}, \quad B_3 = \{0, 2, 5\}, \quad B_4 = \{1, 3, 4\}, \quad B_5 = \{0, 3, 4\}, \quad B_6 = \{1, 2, 5\}. \]

Now, \(B_1\) and \(B_2\) are SRGD and \(G_2 = \{0, 1\}\) and \(G_3 = \{2, 3\}\), \(G_4 = \{4, 5\}\) respectively.

### III. CONCLUSION

In agricultural experiment when the experimental area, where a SRGD is to be applied, could not provide the sufficient block size, by using such proposed construction-method of SRGD it can reduce the block size.
REFERENCES


AUTHOR’S PROFILE

Ksh. Surjit Singh is a Research Scholar in Department of Statistics, Manipur University, India. He has published 2 research papers in International Journal.