An Induction on Semi-Regular Group Divisible Design

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Abstract: A Group Divisible design \((v, b, r, k, \lambda_1, \lambda_2; m, n)\) is said to be Semi-Regular when \(r-\lambda_1 > 0\) and \(rk - v\lambda_2 = 0\). In this paper it is proposed that starting from Semi-Regular Group Divisible (SRGD) design with \(k = m, \lambda_1 = 0\), many series of Group Divisible (GD) designs are constructed without disturbing its Semi-Regular property. Such design are useful in the formation of different plats of same size reinforced Cement concrete, by having choice of different iron bars of different gauges by civil engineers.

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I. INTRODUCTION

Plackett and Burman [1] have given a series of Orthogonal arrays \([4\lambda, 4\lambda-1, 2, 2]\) for all integral \(\lambda \leq 25\), except \(\lambda = 23\) from which Semi-Regular Group Divisible (SRGD) designs with \(\lambda_1 = 0\) can be constructed. For the existence of Group Divisible (GD) design, Bose and Connor [2] have proposed many relations among the parameters of the design. Using some combinatorial methods and the orthogonal arrays, many series of GD designs are presented in the literature of Bose, Shrikhande and Bhattacharya [3]. Many works are available in the book written by Raghava Rao [5]. Two of many contributions proposed by Kageyama and Tsuji [7] i.e., (i) a GD design is Semi-Regular iff \(k/m\) is an integer and every block contains exactly \(k/m\) treatment(s) from each group of the association scheme and (ii) a GD design is singular iff \(k/n\) is integer and every block contains exactly \(k/n\) groups of the association scheme, are enough to mention. Dey and Nigam [8] has suggested a construction method of a GD design \((v = v'/s, b = b'/s, r = r', k = k', \lambda_1 = 0, \lambda_2 = s\lambda_2'; m = m', n = t)\), starting from another GD design \((v' = m'n', b', r', k', \lambda_1' = 0, \lambda_2' = m'n'; n' = st)\); \((s \geq 2, t \geq 2)\). If the starting design is SRGD design, the resultant design is also so, by maintaining the same block size of the starting design. Kageyama [9], starting from \(\alpha\)-resolvable Balanced Incomplete Block (BIB) design, has constructed \(\alpha\)-resolvable regular GD design.

In the sphere of construction of GD designs, Banerjee and Kageyama [10] have contributed a method of constructing Regular Group Divisible (RGD) design. A series of RGD design \((v' = v-1, b' = r(v - k + 1), r' = (r - 1)(k - 1) + 1, k' = k - 1, \lambda_1' = 1, \lambda_2' = k - 2\) have been proposed by Sastry [11], starting from a BIB design \((v, b, r, k, \lambda = 1)\). In the literature of Kumar [12], a method of construction of GD designs from a given BIB design, using Kronecker products which were introduced by Vartak [4] is presented where the constructed PBIB designs, even though they are less efficient than the parent BIB design, becomes necessary for different combinations of “r” (replications) and “k” (block size).

II. CONSTRUCTION

In the following starting from a given SRGD design \((v = mn, b, r, k = m, \lambda_1 = 0, \lambda_2; m, n)\), the construction of another SRGD design \((v' = (m+1)n, b' = nb, r' = nr, k' = m + 1, \lambda_1' = 0, \lambda_2' = n\lambda_2; m, n)\) follows.

A. Theorem: The existence of a SRGD design \((v = mn, b, r, k = m, \lambda_1 = 0, \lambda_2; m, n)\) implies that of a SRGD design \((v' = (m+1)n, b' = nb, r' = nr, k = m + 1, \lambda_1' = 0, \lambda_2' = n\lambda_2; m, n)\). Augment a new group to the \(n\) given groups of the association scheme, are enough to mention. Dey and Nigam [8] has suggested a construction method of a GD design \((v = v'/s, b = b'/s, r = r', k = k', \lambda_1 = 0, \lambda_2 = s\lambda_2'; m = m', n = t)\), starting from another GD design \((v' = m'n', b', r', k', \lambda_1' = 0, \lambda_2' = m'n'; n' = st)\); \((s \geq 2, t \geq 2)\). If the starting design is SRGD design, the resultant design is also so, by maintaining the same block size of the starting design. Kageyama [9], starting from \(\alpha\)-resolvable Balanced Incomplete Block (BIB) design, has constructed \(\alpha\)-resolvable regular GD design.

Proof:- Let \(\theta_{(i-1)+j}\) be the \(j\)th treatment in the \(i\)th group of the Group Divisible association scheme on which the given SRGD design \((v = mn, b, r, k = m, \lambda_1 = 0, \lambda_2; m, n)\) based on \(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\). That is, \(G_1 = \{\theta_1, \theta_2, \ldots, \theta_n\}\), \(G_2 = \{\theta_{n+1}, \theta_{n+2}, \ldots, \theta_{2n}\}\), \ldots, \(G_m = \{\theta_{(m-1)n+1}, \theta_{(m-1)n+2}, \ldots, \theta_{mn}\}\). Augment a new group \(G_{m+1} = \{\theta_{mn+1}, \theta_{mn+2}, \ldots, \theta_{(m+1)n}\}\) to the \(m\) given groups viz.; \(G_1, G_2, \ldots, G_m\). Further, a new block is given by

\[
B_{ij} = B_i \cup \{\theta_{mn+j}\} \ldots \quad (1)
\]

where \(B_i\) is the \(i\)th block of the given SRGD design, \(i = 1, 2, \ldots, b; j = 1, 2, \ldots, n\).

In addition to the \(mn\) treatments of the parent SRGD design, belonged to \(G_1, G_2, \ldots, G_m\); the \(n\) new elements...
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belonged to \(G_{m+1}\) are incorporated in both in the association scheme and in the resultant design. Thus \(v^*=(m+1)n\). As \(j\) takes the values 1, 2, …, \(b\) and simultaneously \(j\) takes 1, 2, …, \(n\), the total number of new blocks \(B_{ij}\)'s is equal to \(nb\). That is \(b^* = nb\). The block size of the new block is equal to \(||B_{ij}|| = ||B_{ij}\cup\{\theta_{mn+j}\}|| = (m+1)\) as \(\theta_{mn+j} \neq \theta_{(i-1)n+j} ; i = 1, 2, \ldots, m\), where by \(||A||\), it denotes the cardinality of a given set \(A\). Since a treatment \(\theta\), in the parent SRGD design gets replicated \(r\) times, there exist \(r\) blocks, say, \(B_s(\theta)\), containing \(\theta\); \(s = 1, 2, \ldots, r\). By the construction method of new block i.e. \(B_{sj} = B_s(\theta)\cup\{\theta_{mn+j}\}\) defined in (1), all the new blocks \(B_{sj}\)'s contain \(\theta\). Each of the treatments \(\theta_1, \theta_2, \ldots, \theta_{mn}\) from the parent design gets replicated \(rn\) times in the resultant design as \(j = 1, 2, \ldots, n\). As \(j\) takes 1, 2, \ldots, \(n\), the total number of new blocks \(B_{ij}\) containing \(\phi\), is \(b\) i.e., \(m = mn, k = m\) and \(vr = bk\). Then every newly introduced treatment \(\theta_{mn+j}\) gets replicated \(rn\) times. Therefore, \(r^* = nr\).

By the construction method of new block \(B_{ij}\) defined in (1), it is cleared that no two treatments belonged to the same group \(G_i\) (\(i = 1, 2, \ldots, m\)) occur together in any new block \(B_{ij}\) and also no two treatments belonged to the group \(G_{m+1}\) have been incorporated in the new block defined in (1) at the same time. So no two treatments belonged to same groups \(G_1, G_2, \ldots, G_{m+1}\) occur together in the resultant design. Thus \(\lambda_1 = 0\).

For counting the number of times of concurrence of two treatments \(\phi, \varphi\) belonged to two different groups (i.e., \(\phi, \varphi\) do not belong to same group), we classify all possible cases into two exhaustive cases: (i) \(\phi, \varphi\) belong to \(G_1, G_2, \ldots, G_m\), but not to a same group (ii) \(\phi, \varphi\) belong to \(G_1, G_2, \ldots, G_{m+1}\). Under the case (i), there are \(\lambda_2\) blocks, say, \(B_i(\phi, \varphi)\); \(i = 1, 2, \ldots, \lambda_2\), containing \(\phi, \varphi\) together in the parent design. From each of \(B_i(\phi, \varphi)\)'s, we can construct \(n\lambda_2\) blocks, \(B_i(\phi, \varphi) \cup \{\theta_{mn+j}\}\) as blocks of the resultant design; \(j = 1, 2, \ldots, n\). As \(j\) takes 1, 2, …, \(n\), the two treatments \(\phi, \varphi\) belonged to two different groups of \(G_1, G_2, \ldots, G_{m+1}\) get replicated together in \(n\lambda_2\) blocks of the resultant design. Further, under the case (ii) there are \(r\) blocks, say, \(B_s(\theta)\); \(s = 1, 2, \ldots, r\), containing \(\theta \in G_i\) (\(i = 1, 2, \ldots, m\)). By the construction method of new blocks of the resultant design defined in (1), each of all the \(r\) new blocks \(B_s(\theta)\cup\{\varphi\}\); \(s = 1, 2, \ldots, r\), contain \(\theta\) and \(\varphi\) together. Thus in the resultant design, \(\theta\), one of two treatments belonged to either \(G_1, G_2, \ldots, G_m\) and another treatment \(\varphi\) belonged to \(G_{m+1}\), appear together in \(r\) (i.e., \(n\lambda_2\) as the design is

Semi-regular i.e., \(\lambda_2 = rn\lambda_2\). Hence the proof is complete.

A. Remark: An increase of block-size of the parent SRGD design by \(1\), makes an increase of treatment numbers of the parent SRGD design, by \(n\). It privileges the experimenters to increase the treatment number without much affecting the block-size wherever necessary.

Applying \(p\) times the same construction method of new blocks of the resultant SRGD design and the same augmenting method of groups on which of the design bases, a corollary is immediate.

A. Corollary: The existence of a SRGD design \((v=mn, b, k=m, n=2)\), Reference number SR19, Clathworthy [6], the construction of another SRGD design \((v'=m+p)n, b^'=bn, k^'=m+p, \lambda_1^'=0, \lambda_2^'=\lambda_2n^p; m^* = m + p, n^* = n)\).

Using a SRGD design \((v=6, b=8, r=4, k=3, \lambda_1 = 6, \lambda_2 = 2; m=3, n=2)\), Reference number SR39, Clathworthy\(\textit{ibid.}\), is shown as an example of the A. Theorem, at below.

A. Example: The 8 blocks of the given SRGD design \((v=6, b=8, r=4, k=3, \lambda_1 = 0, \lambda_2 = 0; m=3, n=2)\) and the 3 groups of the Group Divisible association scheme on which the given SRGD design bases, are \(B_1 = \{0, 2, 4, \ldots, 4\}, B_2 = \{0, 2, 5\}, B_3 = \{0, 3, 4\}, B_4 = \{0, 3, 5\}, B_5 = \{1, 2, 4\}, B_6 = \{1, 2, 5\}, B_7 = \{1, 3, 4\}, B_8 = \{1, 3, 5\}\) and \(G_1 = \{0, 1\}, G_2 = \{2, 3\}, G_3 = \{4, 5\}\) respectively.

By the construction method of new blocks of the resultant SRGD design and the augmenting method of new groups on which the resultant SRGD design bases, we get the 16 blocks and the 4 groups of the SRGD design \((v'=8, b^*=16, r^*=8, k^*=4, \lambda_1^*=0, \lambda_2^*=4, m^*=4, n^*=2)\) as given below.

\[
\begin{align*}
B_1 &= \{0, 2, 4, 6\}, B_2 = \{0, 2, 5, 6\}, B_3 = \{0, 3, 4, 6\}, B_4 = \{0, 3, 5, 6\}, B_5 = \{1, 2, 4, 6\}, B_6 = \{1, 2, 5, 6\}, B_7 = \{1, 3, 4, 6\}, B_8 = \{1, 3, 5, 6\}, B_9 = \{0, 2, 4, 7\}, B_{10} = \{0, 2, 5, 7\}, B_{11} = \{0, 3, 4, 7\}, B_{12} = \{0, 3, 5, 7\}, B_{13} = \{1, 2, 4, 7\}, B_{14} = \{1, 2, 5, 7\}, B_{15} = \{1, 3, 4, 7\}, B_{16} = \{1, 3, 5, 7\}\); \(G_1 = \{0, 1\}, G_2 = \{2, 3\} G_3 = \{4, 5\}, G_3 = \{6, 7\}\).
\]

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