Random Euler Method for Solving a System of Random Matrix Differential Initial Value Problem in Mean Square Sense

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Abstract—In this paper, the random Euler method (REM) is used in solving system of random matrix differential initial value problems of first order. The existence and uniqueness theorem was proved. The REM is presented and the conditions for the mean square (m.s.) convergence are established. Numerical examples show that REM gives good results where some statistical properties of the numerical solutions are computed through numerical case studies.

Keywords: Random Differential Equations (RDE’s), Mean Square Sense (m.s.), Matrix initial Value Problems, Random Euler Method (REM).

I. INTRODUCTION

Random differential equations (RDE’s) are useful to model problems involving rates of changes of quantities representing variables under uncertainties or randomness, being in fact stochastic processes instead of deterministic functions [1], [2], [3], [4], [5] and [6]. The concept of mean square calculus is applied to study system of RDE’s. The REM is used to obtain an approximate solution for equation (1.1):

$$X(t) = F(X(t), t), \quad t \in T \}$$

$$X(t_0) = X_0$$

where $X_0$ is a second order random matrix of size $r \times s$ and, the unknown $X(t)$ is a second order random matrix (2-r.m.) of size $r \times s$ and $F(X(t), t)$ are matrix stochastic processes of size $r \times s$, $T = [t_0, t_e]$. This paper is organized as follows. In section 2, we deal with some preliminary definitions, results, notations and examples. Section 3, is addressed to the proof of existence and uniqueness theorem. Section 4, is addressed to the presentation and the proof of the convergence for the random Euler Scheme in mean square sense. In Section 5, some statistical properties for the exact and numerical solutions are studied. The last section is devoted to conclusions.

II. PRELIMINARIES

Definition2.1. [7]

If $\{X^{ij}: 1 \leq i \leq r, 1 \leq j \leq s \}$ is the set of $r \times s$-r.v.’s, then the second order random matrix (2-r.m.) associated to this family is defined as

$$X = \begin{bmatrix} X^{11} & \cdots & X^{1s} \\ \vdots & \ddots & \vdots \\ X^{r1} & \cdots & X^{rs} \end{bmatrix}$$

(1.2)

The set of all 2-r.m.’s $X$ of size $r \times s$ endowed with the norm

$$\|X\|_{r.m.} = \max_{1 \leq i \leq r, 1 \leq j \leq s} \|X^{ij}\|,$$  

(1.3)

Has a Banach space structure, denoted by $L^r_s$. Where using the notation introduced at (1.2),

$$E[X] = (E[X^{ij}])_{r.m.}.$$  

The m.s. continuity, m.s. differentiation, and m.s. integration associated with a second –order matrix stochastic process are defined with respect to the norm $\|X\|_{r.m.}$.

Definition2.2. [7]

A sequence of 2-r.m.’s $\{X_n\}_{n \geq 0}$ is mean square (m.s.) convergent to $X \in L^r_s$, and will be denoted by $X_n \rightarrow M_X$ as $n \rightarrow \infty$. if:

$$\lim_{n \rightarrow \infty} \|X_n - X\|_{r.m.} = 0.$$  

(1.4)

Theorem2.1. [7]

if $\{X_n\}_{n \geq 0}$ is a sequence of random matrices in $L^r_s$ m.s. convergent to $X$, then:

$$E[X_n] \rightarrow E[X].$$  

(1.5)

Definition2.3. [7]

A 2-m.s. $\{X(t): t \in T\}$ in $L^r_s$ is m.s. continuous at $t \in T$, $T$ an interval of real line, if

$$\lim_{t \rightarrow t_0} \|X(t + \tau) - X(t)\|_{r.m.} = 0,$$  

(1.6)

Definition2.4.[7]

A 2-m.s. $\{X(t): t \in T\}$ in $L^r_s$ is m.s. differentiable at $t \in T$, if there exists a 2-m.s. denoted by $\hat{X}(t): t \in T$ such that:

$$\lim_{\tau \rightarrow 0} \frac{\|X(t + \tau) - X(t)\|_{r.m.}}{\tau} = 0,$$  

(1.7)

Example2.1.[8]

Let $Y$ be a 2-r.m. and let us consider the 2-s.p. $Y(t) = Y(t)$ for $t$ lying in the interval $T$ and applying the formula of integration by parts for $h(t, u) \equiv 1$ one gets:

$$\int_{t_0}^{T} Y \ du = (t - t_0).$$
Proposition 2.1. [8]
If \( \{\mathbf{X}(t), t \in T\} \) is a 2-s.p. m.s. Continuous on \( T = [t_0, t] \), then:
\[
\left\| \int_{t_0}^{t} \mathbf{X}(u) \, du \right\| \leq \int_{t_0}^{t} \left\| \mathbf{X}(u) \right\| \, du \leq M_X(t-t_0), \quad M_X = \max_{T \times T} \left\| \mathbf{X}(u) \right\|.
\] (1.8)

Definition 2.5[7]
Consider equation (1.1) where:
\[
\mathbf{X}: S \times T \to L^2_{s(\infty)} \quad \text{with} \quad S \subseteq L^2_{s(\infty)}
\] is continuously and \( \mathbf{X}_0 \in L^2_{s(\infty)} \). The m.s. \( \mathbf{X}(t): T \to L^2_{s(\infty)} \) is called the mean square (m.s.) solution of (3.1) on \( S \times T \) if:
1. \( \mathbf{X}(t) \) is m.s. continuous on \( T \),
2. \( \mathbf{X}((t_0)) = \mathbf{X}_0 \),
3. \( \mathbf{X}(X(t), t) \) is the m.s. derivative of \( \mathbf{X}(t) \) on \( T \).

Theorem 2.2.
\( \mathbf{X}(t): T \to L^2_{s(\infty)} \) is a m.s. solution of (1.1) if and only if, for all \( t \in T \),
\[
\mathbf{X}(t) = \mathbf{X}_0 + \int_{t_0}^{t} \mathbf{F} (\mathbf{X}(s), s) \, ds \quad \text{(1.9)}
\]
where the integral is understood to be m.s. integral.

III. THE EXISTENCE AND UNIQUENESS

Theorem 3.1.
In equation (1.1), if \( \mathbf{F}: S \times T \to L^2_{s(\infty)} \) satisfies the m.s. Lipschitz condition:
\[
\left\| \mathbf{F}(\mathbf{X}, t) - \mathbf{F}(\mathbf{Y}, t) \right\|_{s(\infty)} \leq k(t) \left\| \mathbf{X} - \mathbf{Y} \right\|_{s(\infty)} \quad \text{(1.10)}
\]
where \( \int_{t_0}^{t} k(t) \, dt < \infty \),
then there exists a unique m.s. solution for any initial condition \( \mathbf{X}_0 \in L^2_{s(\infty)} \).

Proof
The existence can be proved by using successive approximations. Let:
\[
\mathbf{X}_n^1 = \mathbf{X}_0 + \int_{t_0}^{t} \mathbf{F}(\mathbf{X}_n^{n-1}, s) \, ds \quad \text{(1.12)}
\]
for \( n > 1 \)
\[
\mathbf{X}_n^1 = \mathbf{X}_0 + \int_{t_0}^{t} \mathbf{F}(\mathbf{X}_n^{n-1}, s) \, ds \quad \text{(1.12)}
\]
And for \( n = 1 \) we have:
\[
\left\| \mathbf{X}_1 - \mathbf{X}_0 \right\|_{s(\infty)} = \left\| \int_{t_0}^{t} \mathbf{F}(\mathbf{X}(s), s) \, ds \right\|_{s(\infty)} \leq c \left\| \mathbf{e}_{t_0} \right\|
\]
where \( \mathbf{F}(\mathbf{X}(t), t) \). For \( n > 1 \), we have:
\[
\left\| \mathbf{X}_n - \mathbf{X}_n^{n-1} \right\|_{s(\infty)} = \left\| \int_{t_0}^{t} [ \mathbf{F}(\mathbf{X}_n^{n-1}, s) - \mathbf{F}(\mathbf{X}_n^{n-2}, s) ] \, ds \right\|_{s(\infty)} \leq \int_{t_0}^{t} k(s) \left\| \mathbf{X}_n^{n-1} - \mathbf{X}_n^{n-2} \right\|_{s(\infty)} ds
\]
Successively, we can obtain the following:
\[
\left\| \mathbf{X}_n - \mathbf{X}_n^{n-1} \right\|_{s(\infty)} \leq \int_{t_0}^{t} k(s) \left\| \mathbf{X}_n^{n-1} - \mathbf{X}_n^{n-2} \right\|_{s(\infty)} ds \leq \ldots
\]
\[
\leq k^{-1} \left\| \mathbf{X}_n - \mathbf{X}_0 \right\|_{s(\infty)} \leq k \left\| \mathbf{X}_n - \mathbf{X}_0 \right\|_{s(\infty)} \leq \ldots
\]
Therefore,
\[
\left( \mathbf{X}_n - \mathbf{X}_0 \right) \leq \sum_{r=1}^{n} \frac{ck^{-r} \left\| \mathbf{X}_n - \mathbf{X}_0 \right\|_{s(\infty)}}{r!}
\] (1.15)

Note that:
\[
\sum_{r=1}^{\infty} \frac{ck^{-r} \left\| \mathbf{X}_n - \mathbf{X}_0 \right\|_{s(\infty)}}{r!} = \sum_{r=1}^{n} \frac{ck^{-r} \left\| \mathbf{X}_n - \mathbf{X}_0 \right\|_{s(\infty)}}{r!}
\]

Further,
\[
\lim_{n \to \infty} \left( \mathbf{X}_n - \mathbf{X}_0 \right) = \left( \mathbf{X} - \mathbf{X}_0 \right) \quad \text{and only if, for } t \in T,
\]
then the mean square limit i.e. \( \lim_{n \to \infty} \mathbf{X}_n \) exists, i.e. \( \mathbf{X} \). Then the mean square limit i.e. \( \lim_{n \to \infty} \mathbf{X}_n \) exists, since \( \mathbf{X}_0 \) is the general solution of equation (1.12) and \( \mathbf{X} \) is the general solution of equation (1.9).

To prove the uniqueness of the solution, let \( \mathbf{X} \) is a solution of the initial value problem (1.1) and \( \mathbf{Y} \) is the solution of
\[
\begin{align*}
\mathbf{Y}(t) &= \mathbf{F}(\mathbf{Y}(t), t), \quad t_0 < t < t_e \quad \text{(1.9)}
\end{align*}
\]
and for
\[
\mathbf{Y}(t) = \mathbf{Y}_0 + \int_{t_0}^{t} \mathbf{F}(\mathbf{Y}(s), s) \, ds \quad \text{(1.20)}
\]
To complete the proof we need to prove that:
\[
\mathbf{X}(t) = \mathbf{Y}_t \quad \text{(1.21)}
\]
By subtracting (1.20) from (1.9), we obtain:
\[
\mathbf{X}(t) - \mathbf{Y}_t = \mathbf{X}_0 - \mathbf{Y}_0 + \int_{t_0}^{t} \left[ \mathbf{F}(\mathbf{X}(s), s) - \mathbf{F}(\mathbf{Y}(s), s) \right] \, ds
\]
Since \( \mathbf{X}_0 = \mathbf{Y}_0 \), then:
\[
\left\| \mathbf{X}(t) - \mathbf{Y}_t \right\|_{s(\infty)} \leq \int_{t_0}^{t} k(s) \left\| \mathbf{X}_0 - \mathbf{X}_0 \right\|_{s(\infty)} ds \quad \text{(1.23)}
\]
Or \( \mathbf{U}_t \leq k \int_{t_0}^{t} \mathbf{U}_s ds \quad \text{(1.24)} \)
where \( \mathbf{U}_t = \left\| \mathbf{X}(t) - \mathbf{Y}_t \right\|_{s(\infty)} \). From equation (1.24) we have:
\[
\left\| \mathbf{U}_t \right\|_{s(\infty)} \leq k \int_{t_0}^{t} \left\| \mathbf{U}_s \right\|_{s(\infty)} ds = k \left( t - t_0 \right) \left\| \mathbf{U}_t \right\|_{s(\infty)} \quad \text{(1.25)}
\]
and for \( t > t_0 \), then \( k \) must be satisfied the condition:
\[
k \geq 1 \quad \text{(1.26)}
\]
The inequality (1.26) is contradiction because \( k \) must be an independent free constant, hence the only solution of the integral equation (1.24) is \( \mathbf{U}_t = 0 \) or \( \mathbf{X}(t) = \mathbf{Y}_t \quad \text{(1.27)} \)

i.e., the solution of equation (1.1) exists and is unique.

Example 3.1
Consider the linear random vector differential equation
\[
\mathbf{X}(t) = \mathbf{A}(t) \mathbf{X}(t) - \mathbf{Y}(t), \quad t \geq 0; \quad \mathbf{X}(0) = \mathbf{X}_0
\]
where \( \mathbf{A}(t) = e^{-t} \) is a deterministic and continuous function, \( \mathbf{Y}(t) \) is a vector second-order s.p. and \( \mathbf{X}_0 \in L^2_{s(\infty)} \). Show that, by means of theorem 3.1., the mean-square solution of the equation above exists and is unique.

Proof
Since \( \mathbf{F}(\mathbf{X}, t) = \mathbf{A}(t) \mathbf{X}(t) - \mathbf{Y}(t) \)
By substituting in (1.3), we get
\[ ||F(X, t) - F(Y, t)||_n = ||A(t)(X(t) - Y(t)) + Y(t)||_n \]
\[ = ||A(t)X(t) - A(t)Y(t)||_n \leq ||A(t)||_n ||X - Y||_n \]

Since, \( A(t) = e^{-t} \) is a deterministic and continuous function, \( t \geq 0 \), then:
\[ \int_0^\infty A(t)dt = \int_0^\infty e^{-t}dt < \infty \]
then \( F(X, t) = A(t)X(t) - Y(t) \) satisfy the m.s. Lipschitz condition. So the mean-square solution of the equation above exists and is unique.

**Definition 3.1. [5]**

Given a matrix: \( A = (a^{ij}) \) in \( \mathbb{R}^{r \times r} \), we denoted by:
\[ ||A||_\infty \]

the norm defined as:
\[ ||A||_\infty = \max_{1 \leq i \leq r} \sum_{j=1}^{r} |a^{ij}|. \]  

(1.28)

The following lemma deals with the norm of the product of the deterministic matrix function by a random matrix of compatible sizes.

**Lemma 3.1. [8]**

If \( A, B \) be matrices in \( \mathbb{R}^{r \times r}, \mathbb{R}^{s \times s} \) respectively, and \( X, Y \in \mathbb{R}^{r \times s} \), then:
\[ ||AX||_{r \times s} \leq ||A||_{\infty} ||X||_{r \times s}, ||Y||_{r \times s} \]
\[ \leq ||A||_{\infty} ||Y||_{r \times s}. \]  

(1.29)

**Definition 3.2. [8]**

Let \( S \) be a bounded set in \( L_{r \times s}^2 \), an interval \( T \subseteq \mathbb{R} \) and \( h > 0 \), we say that \( F: S \times T \rightarrow L_{r \times s}^2 \) is randomly bounded time uniformly continuous in \( S \) if:
\[ \lim_{h \rightarrow 0} \omega(S, h) = 0, \]  

(1.30)

where
\[ \omega(S, h) = \sup_{x \in S} \sup_{h 
This is the theoretical solution of matrix problem (1.1) then each of its entries function \( X^{ij}(t) \) are m.s. differentiable and applying the m.s. fundamental theorem of calculus in the interval \( [t_n, t_{n+1}] \subseteq [t_0, t_e] \), and considering the \( (i,j) \)-th entry of \( X(u) = F(X(u), u) \), it follows:
\[ X^{ij}(t_{n+1}) - X^{ij}(t_n) = \int_{t_n}^{t_{n+1}} F^{ij}(X(u), u)du \]  

(1.34)

Taking into account the \( (i,j) \)-th entry of (1.32) and example (2.1.) with \( Y = F^{ij}(X_n, t_n) \) one gets:
\[ X^{ij}_{n+1} - X^{ij}_n = hF^{ij}(X_n, t_n) = \int_{t_n}^{t_{n+1}} F^{ij}(X_n, t_n)du. \]  

(1.35)

From (1.33)-(1.35) it follows:
\[ \sum_{1 \leq i \leq r} \sum_{j=1}^{r} \lvert \Delta X^{ij}_n \rvert \leq h \sum_{1 \leq i \leq r} \sum_{j=1}^{r} \lvert F^{ij}(X_n, t_n) \rvert. \]  

(1.36)

In order to prove that the m.s. convergence of the error random matrix sequence \( e_n = (e^{ij}_n) \) to the zero matrix in \( L_{r \times s}^2 \), as \( F(X, t) \) is m.s. continuous, by proposition (2.1.) it follows that
\[ \sum_{1 \leq i \leq r} \sum_{j=1}^{r} \lvert \Delta X^{ij}_n \rvert \leq h \sum_{1 \leq i \leq r} \sum_{j=1}^{r} \lvert F^{ij}(X_n, t_n) \rvert. \]  

(1.37)

**IV. CONVERGENCE OF RANDOM EUCLER METHOD FOR SYSTEM OF RANDOM DIFFERENTIAL EQUATIONS IN MEAN SQUARE SENSE**

The corresponding random matrix Euler method associated to (1.1) takes the form:
\[ X^{i(i+1)}_{n+1} = X^{i(i+1)}_n + hF(X_n, t_n), n \geq 0 \]
\[ X^{i(i)}_0 = X(t_0), \]  

(1.32)

where \( X^{i(i)} \) is the \( (i,j) \)-th entry of the random matrix \( X_n \) and \( F^{i(i)}(X_n, t_n) \) is the \( (i,j) \)-th entry of \( F(X_n, t_n) \). The discretization error \( e_n \) has its corresponding \( (i,j) \)-th entry
\[ e^{i(i)}_n = X^{i(i)}_n - F^{i(i)}(t_n). \]  

(1.33)

We assume the same hypotheses C1 and C2 with only different that the norm in \( L_{r \times s}^2 \) is \( ||.||_{r \times s} \) introduced in equation (1.3). \( F: S \times T \rightarrow L_{r \times s}^2 \) with \( S \subseteq L_{r \times s}^2 \), satisfies:
\[ -C1: F(X, t) \]  

is m.s. randomly bounded time uniformly continuous.

\[ -C2: F(X, t) \]  

Satisfies the m.s. Lipschitz condition:
\[ ||F(X, t) - F(Y, t)||_{r \times s} \leq k(t)||X - Y||_{r \times s} \]
\[ \int_{t_0}^{t_e} k(t)dt < \infty. \]

**Theorem 4.1.**

Under the hypothesis C1 and C2, random matrix Euler method (1.32) is m.s. convergent.
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\( F(X(t_n), t_n) = F^{ij}(X^{ij}(t_n), \ldots, X^{rs}(t_n); \ldots; X^{ij}(t_n), \ldots, X^{rs}(t_n); t_n), \)

\( F(X(t), u), t_n) = F^{ij}(X^{ij}(u), \ldots, X^{ij}(u); \ldots; X^{ij}(u), \ldots, X^{ij}(u); t_n), \)

and by decomposing the difference \( F^{ij}(X(t_n), t_n) - F^{ij}(X(u), t_n) \) in \( r \times s \) terms and applying the Lipschitz condition we have

\[ \| F^{ij}(X(t_n), t_n) - F^{ij}(X(u), t_n) \| \leq \max_{s \leq i \leq S} \sum_{j=1}^{s} \| X^{ij}(u) - X^{ij}(t_n) \|, \]

(1.41) Because:

\[ \| F^{ij}(X(t_n), t_n) - F^{ij}(X(u), t_n) \| \leq \| F(X(t_n), t_n) - F(X(u), t_n) \| \]

\[ \| F(X^{ij}(u), \ldots, X^{ij}(u); \ldots; X^{ij}(u), \ldots, X^{ij}(u); t_n) \| \leq \sum_{j=1}^{s} \| X^{ij}(u) - X^{ij}(t_n) \|, \]

(1.42) As each entry \( X^{ij}(t) \) of the solution \( X(t) \) of problem (1.1) is m.s. differentiable, in an analogous way to (3.13)-(3.14) as in [8] it follows that:

\[ \| X^{ij}(u) - X^{ij}(t_n) \| \leq \int_{t_n}^{u} \| X^{ij}(v) \| dv \leq h M_X, \]

(1.43) where \( X(t) \) is the theoretical solution of problem (1.1) and we have applied that:

\[ \| X^{ij}(t) \| \leq M_X = \sup \{ \| X(t) \|_{r \times s} : t_n \leq t \leq t_e \} \]

(1.44) By subtracting (1.43) from (1.43) one get:

\[ \| F^{ij}(X(t_n), t_n) - F^{ij}(X(u), t_n) \| \leq s \delta^k(t_n) h M_X. \]

For \( 1 \leq i \leq r \leq 1 \leq j \leq s \)

Then:

\[ \max_{1 \leq i \leq r} \sum_{j=1}^{s} \| F^{ij}(X(t_n), t_n) - F^{ij}(X(u), t_n) \| \leq \delta^k(t_n) \| e_n \|_{r \times s} + s^2 \delta^k M_X, \]

(1.46) Taking norms in (1.136) and using (1.37), (1.46) one gets:

\[ \| e_{n+1} \|_{r \times s} \leq \| e_n \|_{r \times s} + h \delta^k(t_n) M_X + h \delta^k (M_X, h). \]

Hence:

\[ \| e_{n+1} \|_{r \times s} \leq \| e_n \|_{r \times s} + h \delta^k(t_n) M_X + h \delta^k (M_X, h). \]

At \( h \to 0 \), we have:

\[ e_n(t) \|_{r \times s} \leq \frac{\| e_n(t) \|_{r \times s}}{\sum_{k=1}^{n} \| e_k \|_{r \times s} + h \delta^k(t_n) M_X + h \delta^k (M_X, h).} \]

As \( e_n = 0 \) and \( nh = \tau_n - \tau_n = t - t_0 \), the last expression can be written in the form:

\[ \| e_{n+1} \|_{r \times s} \leq \frac{\| e_n(t) \|_{r \times s}}{\sum_{k=1}^{n} \| e_k \|_{r \times s} + h \delta^k(t_n) M_X + h \delta^k (M_X, h).} \]

(1.47) Finally we find:

\[ \| e_{n+1} \|_{r \times s} \to 0 \text{ as } h \to 0, n \to \infty, nh = t - t_0. \]

Then random matrix Euler method (1.32) is convergent in mean square sense.

V. CASES STUDY

Example:

Consider the linear system of two random differential equations:

\[ \begin{align*}
  x'(t) &= \alpha x + \beta y \\
  y'(t) &= \beta x + \alpha y
\end{align*} \]

Where \( \alpha, \beta \) are independent random variables with joint probability density function of the uniform distribution:

\[ f(\alpha, \beta) = 1, 0 \leq \alpha \leq 1 \text{ and } 0 \leq \beta \leq 1. \]
The exact solution:
\[
\begin{align*}
x(t) &= e^{(\alpha + \beta)t} + e^{(\alpha - \beta)t} \\
y(t) &= e^{(\alpha + \beta)t} - e^{(\alpha - \beta)t}
\end{align*}
\]
Or the exact solution is the system:
\[
X(t) = \begin{bmatrix} e^{(\alpha + \beta)t} + e^{(\alpha - \beta)t} \\ e^{(\alpha + \beta)t} - e^{(\alpha - \beta)t} \end{bmatrix}
\]
Where: \[X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\]

The numerical solution:
By using random matrix Euler method:
\[
X_{n+1} = X_n + hF(X_n, t_n)
\]
Then:
\[
x_{n+1}(t) = x_n(t) + h(\alpha x_n(t) + \beta y_n(t)) \quad \text{and} \quad \quad \quad \quad x(0) = 2 \quad \text{and} \quad y(0) = 0
\]
This system can be written as:
\[
X_{n+1}(t) = AX_n(t)
\]
Where:
\[
A = \begin{bmatrix} 1 + \alpha h & \beta h \\ \beta h & 1 + \alpha h \end{bmatrix} \quad \text{and} \quad \quad X_n(t) = \begin{bmatrix} x_n(t) \\ y_n(t) \end{bmatrix}
\]
We will find the eigenvalues (\(\lambda_1, \lambda_2\)) of the matrix \(A\), and:
\[
|A - \lambda I| = 0
\]
Then:
\[
\begin{bmatrix} 1 + \alpha h - \lambda & \beta h \\ \beta h & 1 + \alpha h - \lambda \end{bmatrix} = 0
\]
Then:
\[
\lambda_1 = 1 + \alpha h + \beta h \quad \frac{\beta h}{1 + \alpha h - \lambda}
\]
Then:
\[
x_n(t) = C_1 \lambda_1^n + C_2 \lambda_2^n
\]
At: \(x(0) = 2\) then \(C_1 + C_2 = 2\)
if we take \(C_1 = C_2 = 1\):
\[
x_n(t) = (1 + \alpha h + \beta h)^n + (1 + \alpha h - \beta h)^n
\]
And:
\[
y_n(t) = C_1 \lambda_1^n + C_2 \lambda_2^n
\]
At \(y(0) = 0\) then \(C_1 + C_2 = 0\)
if we take \(C_1 = 1\) and \(C_2 = -1\):
\[
y_n(t) = (1 + \alpha h + \beta h)^n - (1 + \alpha h - \beta h)^n
\]
Then the numerical solutions of the system are:
\[
\begin{align*}
x_n(t) &= (1 + \alpha h + \beta h)^n + (1 + \alpha h - \beta h)^n \\
y_n(t) &= (1 + \alpha h + \beta h)^n - (1 + \alpha h - \beta h)^n
\end{align*}
\]
Or the numerical solution of the system (*) is
\[
X_n(t) = \begin{bmatrix} (1 + \alpha h + \beta h)^n + (1 + \alpha h - \beta h)^n \\ (1 + \alpha h + \beta h)^n - (1 + \alpha h - \beta h)^n \end{bmatrix}
\]
Verification:
(i) \(\lim_{n \to \infty} X_n(t) = X(t)\)
We will prove that:
\[
\lim_{n \to \infty} x_n(t) = x(t)
\]
And:
\[
\lim_{n \to \infty} y_n(t) = y(t)
\]
First:
\[
\lim_{n \to \infty} x_n(t) = x(t)
\]
Since:
\[
x_n(t) = (1 + \alpha h + \beta h)^n + (1 + \alpha h - \beta h)^n
\]
Then:
\[
\lim_{n \to \infty} x_n(t) = \lim_{n \to \infty} \left( (1 + h(\alpha + \beta))^n + (1 + h(\alpha - \beta))^n \right) = \lim_{n \to \infty} \left( (1 + h(\alpha + \beta))^n + (1 + h(\alpha - \beta))^n \right)
\]
Where:
\[
h = \frac{t_n - t_0}{n} = \frac{t_n - 0}{n} = \frac{t_n}{n}
\]
Then:
\[
\lim_{n \to \infty} x_n(t) = \lim_{n \to \infty} \left( (1 + \frac{t_n}{n}(\alpha + \beta))^n \right) + \lim_{n \to \infty} \left( (1 + \frac{t_n}{n}(\alpha - \beta))^n \right) = e^{(\alpha + \beta)t} + e^{(\alpha - \beta)t} = x(t)
\]
Since:
\[
y_n(t) = (1 + \alpha h + \beta h)^n - (1 + \alpha h - \beta h)^n
\]
Then:
\[
\lim_{n \to \infty} y_n(t) = \lim_{n \to \infty} \left( (1 + h(\alpha + \beta))^n - (1 + h(\alpha - \beta))^n \right) = \lim_{n \to \infty} \left( (1 + h(\alpha + \beta))^n - (1 + h(\alpha - \beta))^n \right)
\]
Where:
\[
h = \frac{t_n - t_0}{n} = \frac{t_n - 0}{n} = \frac{t_n}{n}
\]
Then:
\[
\lim_{n \to \infty} y_n(t) = \lim_{n \to \infty} \left( (1 + \frac{t_n}{n}(\alpha + \beta))^n \right) + \lim_{n \to \infty} \left( (1 + \frac{t_n}{n}(\alpha - \beta))^n \right) = e^{(\alpha + \beta)t} - e^{(\alpha - \beta)t} = y(t)
\]
Hence:
\[
\lim_{n \to \infty} X_n(t) = X(t)
\]
Now:
\[
(i) \lim_{n \to \infty} E\{X_n(t)\} = E\{X(t)\}
\]
We will prove that:
\[
\lim_{n \to \infty} E\{x_n(t)\} = E\{x(t)\}
\]
And:
\[
\lim_{n \to \infty} E\{y_n(t)\} = E\{y(t)\}
\]
First:
\[
\lim_{n \to \infty} E\{x_n(t)\} = E\{x(t)\}
\]
Since:
\[
x_n(t) = (1 + \alpha h + \beta h)^n + (1 + \alpha h - \beta h)^n
\]
\[
= (1 + h(\alpha + \beta))^n + (1 + h(\alpha - \beta))^n
\]
Then:
\[
E\{x_n(t)\} = E \left( \sum_{k=0}^{n} \binom{n}{k} h^k ((\alpha + \beta)^k + (\alpha - \beta)^k) \right)
\]
We will find \(E\{((\alpha + \beta)^k + (\alpha - \beta)^k)\} \) since \(\alpha, \beta \sim U(0, 1)\) and joint PDF \(f(\alpha, \beta) = 1\) then
Random Euler Method for Solving a System of Random Matrix Differential Initial Value Problem in Mean Square Sense

\[ E \{ ((\alpha + \beta)^k + (\alpha - \beta)^k) \} = \int \int \limits_{-\infty}^{\infty} \int \int \limits_{-\infty}^{\infty} ((\alpha + \beta)^k + (\alpha - \beta)^k) f(\alpha, \beta) \, d\beta \, d\alpha \]

\[ = \int \int \limits_{0}^{1} ((\alpha + \beta)^k + (\alpha - \beta)^k) \, d\beta \, d\alpha \]

\[ = \int \limits_{0}^{1} \left\{ \left( \frac{(\alpha + \beta)^{k+1}}{k+1} \right) - \left( \frac{(\alpha - \beta)^{k+1}}{k+1} \right) \right\} \, d\alpha \]

\[ = \frac{1}{(k+1)(k+2)} \left( 1 - \frac{(\alpha + 1)^{k+2}}{k+1} - \frac{(\alpha - 1)^{k+2}}{k+1} \right) \]

Then:

\[ E(x_n(t)) = \sum_{k=0}^{\infty} \left( \frac{n!}{k!} \right) h^k \frac{(2)^{k+2} - 1 + (1)^{k+2} + (-1)^{k+2}}{(k+1)(k+2)} \]

Hence:

\[ \lim_{n \to \infty} E(x_n(t)) = \lim_{n \to \infty} \sum_{k=0}^{\infty} \left( \frac{n!}{k!} \right) h^k \frac{(2)^{k+2} - 1 + (1)^{k+2} + (-1)^{k+2}}{(k+1)(k+2)} \]

Where:

\[ h = \frac{t_n - t_0}{n} = \frac{t_n - t_0}{n} = \frac{t_n}{n} \]

\[ \lim_{n \to \infty} E(x_n(t)) = \sum_{k=0}^{\infty} \left( \frac{n!}{k!} \right) h^k \frac{(2)^{k+2} - 1 + (1)^{k+2} + (-1)^{k+2}}{(k+1)(k+2)} \]

Since:

\[ x(t) = e^{(\alpha+\beta)t} + e^{(\alpha-\beta)t} \]

Then:

\[ E(x(t)) = E \left( \sum_{k=0}^{\infty} \left( \frac{n!}{k!} \right) h^k \frac{(2)^{k+2} + (1)^{k+2}}{(k+1)(k+2)} \right) \]

\[ = \sum_{k=0}^{\infty} \left( \frac{n!}{k!} \right) h^k \frac{(2)^{k+2} + (1)^{k+2}}{(k+1)(k+2)} \]

i.e.

\[ \lim_{n \to \infty} E(x_n(t)) = E(x(t)) \]

Second:

\[ \lim_{n \to \infty} E(y_n(t)) = E(y(t)) \]

Since:

\[ y_n(t) = (1 + ah + bh)^n - (1 + ah - bh)^n \]

\[ = (1 + h(\alpha + \beta))^n - (1 + h(\alpha - \beta))^n \]

\[ = \sum_{k=0}^{\infty} \left( \frac{n!}{k!} \right) h^k ((\alpha + \beta)^k - (\alpha - \beta)^k) \]

Then:

\[ E(y(t)) = \sum_{k=0}^{\infty} \left( \frac{n!}{k!} \right) h^k ((\alpha + \beta)^k - (\alpha - \beta)^k) \]

We will find \( E \{ ((\alpha + \beta)^k - (\alpha - \beta)^k) \} \) since, \( \alpha, \beta \sim U(0,1) \) and joint PDF \( f(\alpha, \beta) = 1 \) then:

\[ E \{ ((\alpha + \beta)^k - (\alpha - \beta)^k) \} = \int \int \limits_{-\infty}^{\infty} ((\alpha + \beta)^k - (\alpha - \beta)^k) \, d\beta \, d\alpha \]

\[ = \int \int \limits_{0}^{1} ((\alpha + \beta)^k - (\alpha - \beta)^k) \, d\beta \, d\alpha \]

\[ = \int \limits_{0}^{1} \left\{ \left( \frac{(\alpha + \beta)^{k+1}}{k+1} \right) - \left( \frac{(\alpha - \beta)^{k+1}}{k+1} \right) \right\} \, d\alpha \]

\[ = \frac{1}{(k+1)(k+2)} \left( 1 - \frac{(\alpha + 1)^{k+2}}{k+1} - \frac{(\alpha - 1)^{k+2}}{k+1} \right) \]

Then:

\[ E(y_n(t)) = \sum_{k=0}^{\infty} \left( \frac{n!}{k!} \right) h^k ((\alpha + \beta)^k - (\alpha - \beta)^k) \]

Hence:

\[ \lim_{n \to \infty} E(y_n(t)) = \lim_{n \to \infty} \sum_{k=0}^{\infty} \left( \frac{n!}{k!} \right) h^k ((\alpha + \beta)^k - (\alpha - \beta)^k) \]

\[ = \sum_{k=0}^{\infty} \left( \frac{n!}{k!} \right) h^k ((\alpha + \beta)^k - (\alpha - \beta)^k) \]

Since:

\[ y(t) = e^{(\alpha+\beta)t} - e^{(\alpha-\beta)t} \]

\[ = \sum_{k=0}^{\infty} \frac{k^k}{k!} (\alpha + \beta) - \sum_{k=0}^{\infty} \frac{k^k}{k!} (\alpha - \beta) \]

Then:

\[ E(y(t)) = E \left( \sum_{k=0}^{\infty} \frac{k^k}{k!} (\alpha + \beta) - \sum_{k=0}^{\infty} \frac{k^k}{k!} (\alpha - \beta) \right) \]

\[ = \sum_{k=0}^{\infty} \frac{k^k}{k!} E((\alpha + \beta)^k) - \sum_{k=0}^{\infty} \frac{k^k}{k!} E((\alpha - \beta)^k) \]

\[ = \sum_{k=0}^{\infty} \frac{k^k}{k!} E((\alpha + \beta)^k) - \sum_{k=0}^{\infty} \frac{k^k}{k!} E((\alpha - \beta)^k) \]

i.e.

\[ \lim_{n \to \infty} E(y_n(t)) = E(y(t)) \]

Then:

\[ \lim_{n \to \infty} E(X_n(t)) = E(X(t)) \]
VI. CONCLUSIONS

The initially valued system of first order random differential equations can be solved numerically using the random Euler method in mean square sense. The existence and uniqueness of the solution have been proved. The convergence of the random Euler method has been proved in mean square sense. The results of the paper have been illustrated through some examples.

REFERENCES


